# Density of states and growth-rate eigenvalue separation statistics of the ideal interchange mode spectrum

R.L. Dewar[1*], C. Nührenberg[2], T. Tatsuno[3]

[1]*Department of Theoretical Physics, RSPhysSE, The Australian National University, Canberra ACT 0200, Australia*
[2]*Max-Planck-Institut für Plasmaphysik, Teilinstitut Greifswald, D-17491 Germany*
[3]*Institute for Research in Electronics and Applied Physics, University of Maryland, College Park, MD 20742-3511, USA*

The ideal-MHD spectrum of normal modes is difficult to characterize mathematically because the Hermitian operators $L$ and $M$ in the generalized eigenvalue equation $L\varphi = \lambda M\varphi$ are unbounded (where $\varphi$ is a stream function in the reduced MHD approximation and the eigenvalue $\lambda \equiv \omega^2 < 0$ for instability). In three-dimensional geometry (stellarators) this is further complicated by the coupling of different poloidal ($m$) and toroidal ($n$) Fourier harmonic. It is the purpose of this paper to explore whether the statistical approach used in quantum chaos theory [1] provides a suitable framework for characterizing the MHD normal mode spectrum, and whether this spectrum falls in the same universality class as the Schrödinger spectrum for a system with a comparable degree of symmetry.

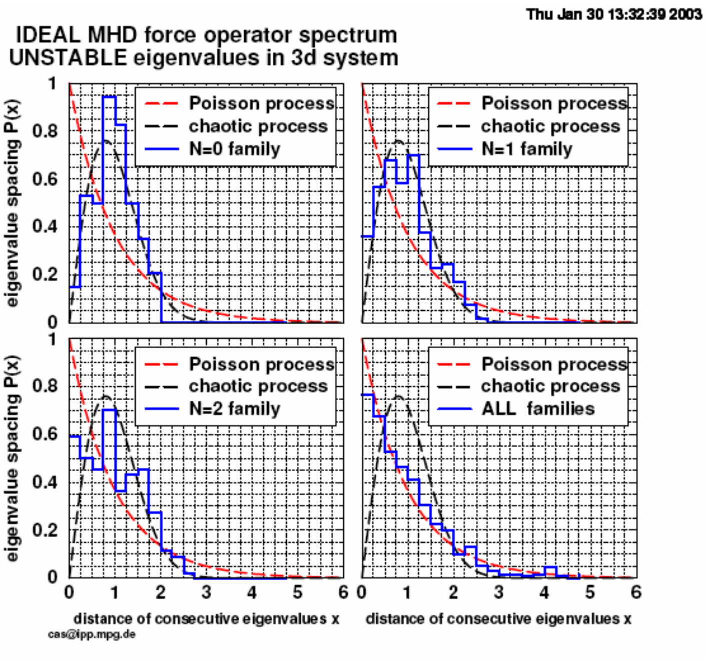


**Figure 1.** Probability distribution function for the spacing of nearest-neighbour unstable eigenvalues, normalized so the average separation is unity, for a Mercier-unstable W7-X variant equilibrium [4], obtained using CAS3D.

The primary motivation for this is the substantial investment in effort in developing numerical matrix eigenvalue programs such as the three-dimensional TERPSICHORE [2] and CAS3D [3] codes. The eigenvalue statistics of unstable interchange modes in a stellarator, Fig. 1, indicates the $N = 0$ mode family in particular to have a distribution close to that expected for a quantum-chaotic system, while the statistics obtained by mixing all three mode families is close to the Poisson distribution characteristic of generic integrable systems. Also, an alternative approach using the ballooning representation and semiclassical (WKB) quantization to estimate the global MHD spectrum has indicated the generic occurrence of chaotic [5,6] ray trajectories.

The present paper discusses the integrable limit, a system with a sufficient number of symmetries to make the ray Hamiltonian integrable and the eigenvalue problem separable. The geometry is the circular cylinder, periodic in the $z$-direction to make it topologically toroidal—we shall refer to the $z$-direction as the toroidal direction and the azimuthal, $\theta$-direction as the poloidal direction. We study a plasma in which the Suydam criterion for the stability of interchange modes is violated, so the number of unstable modes is infinite. Figure 2 shows the level curves of constant growth rate plotted (a) in $m,n$ space and (b) in $\mu,m$ space, where $\mu = n/m$, for a test case analyzed in more detail in [7]. The eigenvalues (dots) are selected by the quantization conditions that $m$ and $n$ be integers. As $m$ and $n$ approach infinity, keeping $\mu$ fixed, the growth-rate eigenvalues asymptote to a constant, the Suydam growth rate, depending only on $\mu$ and the radial mode number $l$ (= 0 in Fig. 2).

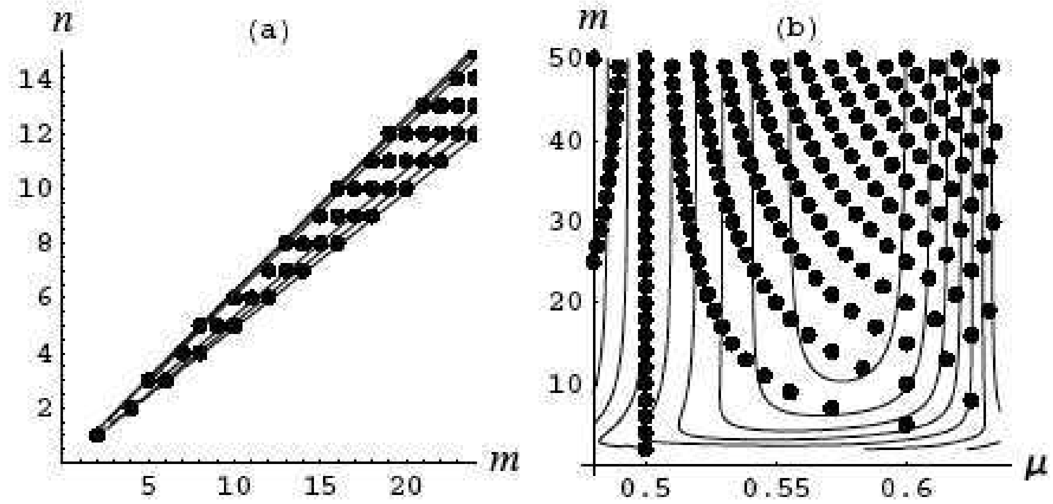


**Figure 2.** Lattice of poloidal and toroidal quantum and growth rate level curves up to a cutoff value $m_{\max}$, for the test case discussed in [7],

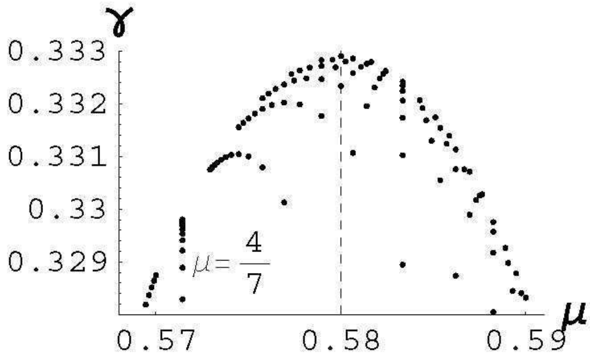


**Figure 3.** Growth rate versus $\mu = n/m$ showing the importance of $1/m^2$ corrections and the gap around the low-order rational $\mu = 4/7$.

The average separation of eigenvalues scales as $1/m^2_{\max}$ and the smallest correction to the $m = \infty$, Suydam eigenvalue also scales as $1/m^2_{\max}$, so these corrections are always important no matter how high we take $m_{\max}$, as is seen in Fig. 3. Thus the eigenvalue separation probability distribution calculated using the Suydam approximation [8] does not reflect the true statistics of interchange modes.

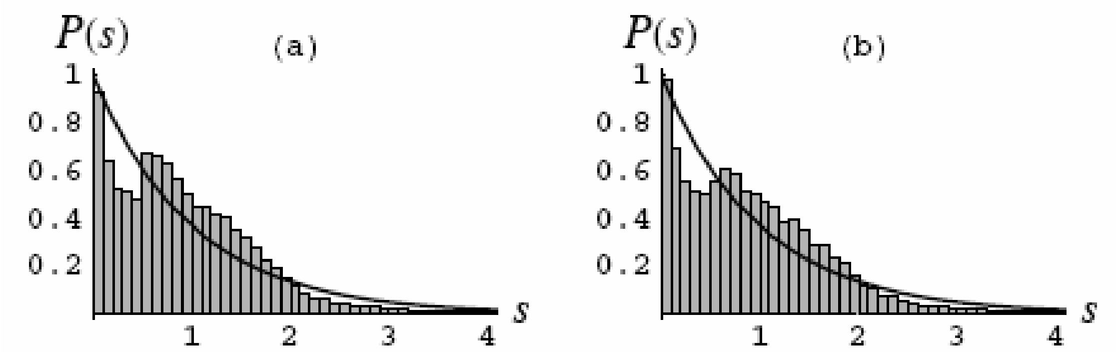


**Figure 4.** Probability distribution function for the spacing of nearest-neighbour unstable eigenvalues, normalized so the average separation is unity, for modes with $\mu$ (a) less than and (b) greater than that corresponding to the most unstable eigenvalue (see Fig. 3).

The eigenvalue separation statistics shown in Fig. 4 show considerable departure from the Poisson statistics expected for a generic integrable wave system, reflecting the considerable number-theoretic structure in the spectrum revealed in Fig. 3. This shows that caution must be applied in interpreting the three dimensional stellarator results shown in Fig. 1 in terms of conventional quantum chaos theory.

*Work supported by the Australian Research Council

**References**

[1] F. Haake, Quantum Signatures of Chaos, 2nd. Ed. (Springer-Verlag, berlin, 2001)

[2] D.V. Anderson et al., Int. J. Supercomp. Appl. **4**, 34 (1990)

[3] C. Schwab, Phys. Fluids B **5**, 3195 (1993).

[4] C. Nührenberg, Phys. Plasmas **3**, 2401 (1996)

[5] R.L. Dewar, P. Cuthbert and R. Ball, Phys. Rev. Letters **86**, 2321 (2001)

[6] M.H. Redi, J.L. Johnson, S. Klasky, J. Canik, R.L. Dewar, and W.A. Cooper, Phys. Plasmas **9**, 1990 (2002)

[7] *Statistical characterization of the interchange-instability spectrum of a separable ideal-magnetohydrodynamic model system*, R.L. Dewar, T. Tatsuno, Z. Yoshida, C. Nührenberg, and B.F. McMillan, submitted to Phys. Rev. E, May 2004, preprint at arXiv physics/0405095

[8] *Quantum chaos theory and the spectrum of ideal-MHD instabilities in toroidal plasmas*, Proceedings of the 13th International Toki Conference, Toki, Japan, 9–12 December 2003, accepted for publication in J. Plasma Fusion Res. SERIES 2004