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THEORY OF THE TIME-DEPENDENT CHILD-LANGMUIR SHEATH

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For simplicity, the plasma-wall transition layer is usually split into a presheath and a sheath region [1], and the sheath (especially for highly negative wall potentials) is further subdivided into two regions, namely the *Langmuir-Debye sheath,* where the electron density must still be taken into account (but drops rapidly), and the electron-free *Child-Langmuir sheath, where* the electron density is practically negligible [2]. In fact, the presheath-sheath interface, the *sheath edge,* represents the transition from the quasineutral presheath to the Langmuir-Debye sheath.

In the present paper we show that the problem of the time-dependent Child-Langmuir sheath can be solved in quite general form, and that it only remains to apply the general solution to a given particular physical situation. In an application of our general result, we consider the limiting case when he wall potential changes adiabatically.

We assume a collision-free sheath in front of an absorbing plane wall located at $z = 0$. The plasma, consisting of electrons and singly charged ions, occupies the half space $z < 0$. The ions are described by the hydrodynamic continuity and momentum equations

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial z} nu = 0, \qquad \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial z} = \frac{e}{m} E, \tag{1}$$

with the electric field $E = -\partial\varphi/\partial z$ satisfying Poisson's equation and Ampère's law,

$$\frac{\partial E}{\partial z} = 4\pi e \left(n - n_e\right), \qquad \frac{\partial E}{\partial t} + 4\pi \left(enu + J_e\right) = 0. \tag{2}$$

Here, n and u are ion density and velocity, m is the ion mass, e is the magnitude of the electron charge, φ is the electric potential, and n_e and J_e are the electron density and the electron current, respectively. Inserting n from Poisson's equation. into Ampère's law we find

$$\frac{\partial E}{\partial t} + u\frac{\partial E}{\partial z} = -4\pi(J_e + en_e u). \tag{3}$$

Further we assume that the ion velocity is much smaller than the electron one, $u \ll u_e = |J_e|/en_e$, and hence can be neglected, The main equations than acquire the form

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial z} = \frac{e}{m}E, \qquad \frac{\partial E}{\partial t} + u\frac{\partial E}{\partial z} = -4\pi J_e. \tag{4}$$

In what follows, we assume that the time dependence of the problem is dictated by the ion dynamics, so that the characteristic time and space scales of the problem, $\Delta\tau$ and Δl, satisfy the inequality $V_{Te} \gg \Delta l/\Delta\tau$, where $V_{Te} = \sqrt{T_e/m_e}$ is the electron thermal velocity. For the electron distribution function we use the general solution of the Vlasov equation,

$$f_e = \bar{f}_e\left(\frac{m_e v^2}{2} - e\varphi(z,t)\right)\Theta\left(\sqrt{\frac{2e}{m_e}\{|\varphi_w(t)| - |\varphi(z,t)|\}} + v\right), \tag{5}$$

where $\varphi_w(t)$ $(\leqslant 0)$ is the time-dependent negative potential on the wall, $\bar{f}_e$ is an arbitrary function of its argument, and $\Theta(x)$ is the Heaviside step function describing the cut-off due to electron absorption at the wall. To be specific, we choose for $\bar{f}_e(s)$ the Maxwell-Boltzmann distribution function, $\bar{f}_e(s) \propto \exp\left(-s/T_e\right)$. Then for the electron current we obtain

$$J_e(t) = -\sqrt{\frac{2}{\pi}}en_0 V_{Te}\frac{\exp\left(-\frac{e}{T_e}|\varphi_w(t)|\right)}{1 + erf\left\{\sqrt{\frac{e}{T_e}|\varphi_w(t)|}\right\}}, \tag{6}$$

where n_0 is the unperturbed electron density, $n_e(z,t) \to n_0$ and $\varphi(z,t) \to 0$ for $z \to -\infty$, and $erf(x)$ is the error function.

Note that the electron current (6) depends only on time. By means of the variable transformation $z = z(E,t)$ and $u(z,t) = u(E,t)$ we find the general solution of Eqs. (4) in the form

$$\begin{aligned} z &= z(E,t) = \frac{e}{m}\int^t dt' \int^{t'} dt'' \left\{E + 4\pi\int_{t''}^{t} dt''' J_e(t''')\right\} + \\ &+ t\cdot\Phi_1\left(E + 4\pi\int^t dt' J_e(t')\right) + \Phi_2\left(E + 4\pi\int^t dt' J_e(t')\right), \end{aligned} \tag{7}$$

where $\Phi_1(x)$ and $\Phi_2(x)$ are arbitrary functions which must be defined according to the initial and boundary conditions of the concrete problem considered. However, the absence of a sharp sheath edge and the ignorance of accurate boundary (or initial) conditions for the Child-Langmuir sheath compel us to make simplifications at the definition of these arbitrary functions. When formulating the Child-Langmuir law, the electric field and the ion velocity at the entrance of the electron-free sheath are usually neglected due to their smallness in comparison with their values inside of the sheath [2, 3, 4]. In the present work, this condition is relaxed by assuming a suitable electric-field value $E_0 = const$, from which on the electron density is neglected. Obviously, the position where $E = E_0$, z_0, will change in time. The interval $(0, z_0(t; E_0))$ we interpret as the width of the Child-Langmuir sheath.

For the further calculations it is convenient to introduce the auxiliary function $F(t)$, defined by the relation $J_e(t) = -\partial F(t)/\partial t$, and the new variable E' instead of t':

$$E' = E + 4\pi F(t) - 4\pi F(t'). \tag{8}$$

The general solution (7) then assumes the form

$$\begin{aligned} z(t) &= z_0\left\{t\left\{E_0 - E - 4\pi F(t)\right\}; E_0\right\} \\ &+\frac{e}{m}\int_{E_0}^{E}\frac{dE'}{4\pi J_e\left\{t'\left\{E' - E - 4\pi F(t)\right\}\right\}}\int_{E_0}^{E'}\frac{E''dE''}{4\pi J_e\left\{t''\left\{E'' - E - 4\pi F(t)\right\}\right\}} \\ &-u_0\left\{t\left\{E_0 - E - 4\pi F(t)\right\}; E_0\right\}\cdot\int_{E_0}^{E}\frac{dE'}{4\pi J_e\left\{t'\left\{E' - E - 4\pi F(t)\right\}\right\}}, \end{aligned} \tag{9}$$

where $t'\left\{E' - E - 4\pi F(t)\right\}$ is the solution of Eq. (8).

For the limiting case of an adiabatically changing wall potential and electron current (6) ($\omega_{pi}\Delta\tau \gg 1$, with ω_{pi} the ion plasma frequency), we can find from (8) the explicit relation

$$t' \simeq t - \frac{E' - E}{4\pi J_e(t)}. \tag{10}$$

Then from (9) we find

$$z - z_0\left\{t; E_0\right\} \simeq \frac{e}{m}\frac{1}{6(4\pi J_e(t))^2}\left\{E^3 + \frac{9}{4}\frac{1}{4\pi J_e^2(t)}\frac{\partial J_e(t)}{\partial t}E^4\right\}, \tag{11}$$

where the second term on the right-hand side describes the small deflection due to the time-dependence of the wall potential, i.e.,

$$\left|E\frac{1}{4\pi J_e^2(t)}\frac{\partial J_e(t)}{\partial t}\right| \approx \left|\frac{E}{4\pi J_e(t)\cdot\Delta\tau}\right| \ll 1. \tag{12}$$

Since solving Eqs (9) and (11) for finite E_0 promises to be quite complicated, we content ourselves here with the usual approximation, $E_0 \to 0$. For the potential in the sheath we then find

$$\begin{aligned} \frac{e}{T_e}\Delta\left|\varphi(z,t)\right| &= \frac{e}{T_e}\left\{\left|\varphi(z,t)\right| - \left|\varphi(z_0,t)\right|\right\} \\ &= 2\left(\frac{3}{4}\right)^{4/3}\left\{\frac{J_e(t)}{en_0 v_s}\right\}^{2/3}\left\{\left(\frac{z-z_0}{\lambda_{De}}\right)^{4/3}\right. \\ &\left. + \frac{6}{5}\left(\frac{3}{4}\right)^{1/3}\frac{1}{\omega_{pi}J_e(t)}\frac{\partial J_e(t)}{\partial t}\left\{\frac{J_e(t)}{en_0 v_s}\right\}^{-1/3}\left(\frac{z-z_0}{\lambda_{De}}\right)^{5/3}\right\}, \end{aligned} \quad (13)$$

where $\varphi(z_0,t)$ is the potential at the sheath entrance.

From (13) follows the remarkable result that for slowly changing wall potential (i. e., $\left|(\omega_{pi}J_e(t))^{-1}\,\partial J_e(t)/\partial t\right| \ll 1$) the Child-Langmuir contains, besides the usual term proportional to $(z-z_0)^{4/3}$, an additional correction term proportional to $(z-z_0)^{5/3}$.

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