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Wavelet representation of plasma particle dynamics in stationary electrostatic waves

S. Tanriverdi and M. Psimopoulos

Plasma Group, Physics Department, Imperial College, London SW7 2BZ, England

The evolution (damping or growing) of electrostatic plasma waves (EPW) is governed by the self- consistent Vlasov-Poisson system [1] which can be solved in the linear limit by using the Landau method [2] or the Van Kampen technique [3]. The nonlinear evolution [4] of EPW is still under investigation [5]. On the other hand, the interaction (exchange of momentum and energy) of a stationary (constant amplitude) EPW with a plasma was studied in the linear case by Dawson [6] and the nonlinear theory was developed by O'Neil [7] using kinetic description of the plasma. In parallel, Stix [8] studied also the linear case as an initial value problem using particle dynamics. In the present work we extend the theory of Stix in the nonlinear case using a wavelet representation of particle dynamics so that the transition from linear to nonlinear regime is manifested.

The motion of a plasma particle in a stationary EPW $E = E_o \cos(kx - \omega t)$ is governed in the wave frame by

$$\frac{dx}{dt} = \upsilon \;\; ; \quad \frac{d\upsilon}{dt} = \varepsilon \cos(kx) \tag{1}$$

where $\varepsilon = qE_o / m$. The *formal solution* of (1) is given by

$$x(t) = x' + \upsilon' t + \varepsilon \int_0^t dt_1 \int_0^{t_1} \cos[kx(t_2)] dt_2 \, ; \quad \upsilon(t) = \upsilon' + \varepsilon \int_0^t \cos[kx(t_1)] dt_1 \tag{2}$$

Using Eqs.(2) we can derive the coefficients $\ddot{b}_i(t)$ of the amplitude expansion of the force exerted on a particle

$$F(t) = m\varepsilon \cos[k(t)] = m\left[\varepsilon \ddot{b}_1(t) + \varepsilon^2 \ddot{b}_2(t) + \varepsilon^3 \ddot{b}_3(t) + \cdots\right] \tag{3}$$

where $\ddot{b}_i(t)$ have the form

$$\ddot{b}_1(t) = \ddot{P}_{11}(\upsilon', t)\cos(kx') + \ddot{Q}_{11}(\upsilon', t)\sin(kx')$$

$$\ddot{b}_2(t) = \ddot{T}_2(\upsilon', t) + \ddot{P}_{22}(\upsilon', t)\cos(2kx') + \ddot{Q}_{22}(\upsilon', t)\sin(2kx')$$

$$\begin{aligned}\ddot{b}_3(t) = {} & \ddot{P}_{31}(\upsilon', t)\cos(kx') + \ddot{Q}_{31}(\upsilon', t)\sin(kx') \\ & + \ddot{P}_{33}(\upsilon', t)\cos(3kx') + \ddot{Q}_{33}(\upsilon', t)\sin(3kx')\end{aligned}$$

$$\ddot{b}_4(t) = \ddot{T}_4(\upsilon',t) + \ddot{P}_{42}(\upsilon',t)\cos(2kx') + \ddot{Q}_{42}(\upsilon',t)\sin(2kx') + \ddot{P}_{44}(\upsilon',t)\cos(4kx') + \ddot{Q}_{44}(\upsilon',t)\sin(4kx')$$

Detailed calculations [9] show that $T_i(\upsilon',t)\,;P_{ij}(\upsilon',t)\,;Q_{ij}(\upsilon',t)$ are *wavelets* with respect to υ' which are well behaved for both $\upsilon'=0$ and $\upsilon'=\to\pm\infty$ and gradually transform into singular integrable functions for $t >> t_o$, where $t_o = (m/k^2T)^{1/2}$ is the phase mixing time.

The force per unit volume exerted on a plasma by a stationary EPW is obtained by averaging the force (3) acting on each plasma particle over initial conditions (x',υ').

$$n_o\langle F(t)\rangle_{x',\upsilon'} = n_o m\left[\varepsilon\langle\ddot{b}_1(t)\rangle_{x',\upsilon'} + \varepsilon^2\langle\ddot{b}_2(t)\rangle_{x',\upsilon'} + \cdots\right] \tag{4}$$

For a homogenous plasma only even power terms remain non-zero after space averaging over wavelength $l = 2\pi k$ so that the above force depends only on the irreducible wavelets $\ddot{T}_2(\upsilon',t)$, $\ddot{T}_4(\upsilon',t)$,...Up to fourth order we have

$$n_o\langle F(t)\rangle_{x',\upsilon'} = n_o m\int_{-\infty}^{+\infty}[\varepsilon^2\ddot{T}_2(\upsilon',t) + \varepsilon^4\ddot{T}_4(\upsilon',t)]\phi_o(\upsilon'-u)d\upsilon' \tag{5}$$

where $u = -\omega/k$ is the plasma fluid velocity in the wave frame. Introducing next $\ddot{T}_2(\upsilon',t) = (\pi/2k)\partial K_2(\upsilon',t)/\partial\upsilon'$ and $\ddot{T}_4(\upsilon',t) = -(\pi k t^4/384)\partial K_4(\upsilon',t)/\partial\upsilon'$ where explicitly

$$K_2(\upsilon',t) = \frac{\sin(k\upsilon' t)}{\pi\upsilon'} \quad ; \quad K_4(\upsilon',t) = \frac{24}{\pi k^4 t^4 \upsilon'^5}\begin{bmatrix}\sin(k\upsilon' t) - 5\sin(2k\upsilon' t) + 7k\upsilon' t\cos(k\upsilon' t) \\ + 2k\upsilon' t\cos(2k\upsilon' t) + k^2\upsilon'^2 t^2\sin(k\upsilon' t)\end{bmatrix}$$

are normalized wavelets [9] that become δ-functions for $t >> t_o$, we get

$$n_o\langle F(t)\rangle_{x',\upsilon'} = -\frac{\pi}{2}n_o\frac{m\varepsilon^2}{k}\int_{-\infty}^{+\infty}\left[K_2(\upsilon',t) - \frac{\varepsilon^2 k^2}{192}t^4 K_4(\upsilon',t)\right]\frac{\partial}{\partial\upsilon'}\phi_o(\upsilon'-u)d\upsilon'$$

$$\approx -\frac{\pi}{2}n_o\frac{m\varepsilon^2}{k}\left[\frac{\partial\phi_o(\upsilon)}{\partial\upsilon}\right]_{\upsilon=\omega/k}\left\{1 - \frac{k^2\varepsilon^2}{192}t^4 + \cdots\right\} \tag{6}$$

The first term of this result gives the linear part of the force whereas the second term describes the transition from linear to nonlinear regime. Clearly nonlinear effects reduce the linear force by a *universal function* $M(\tau) \approx 1 - (\tau^4/192) + O(\tau^8)$ which is independent of the velocity distribution and depends only on $\tau = t/t_{osc}$ where $t_{osc} = (k\varepsilon)^{-1/2}$ is the trapped particle oscillation period.

The energy rate per unit volume transferred to the plasma by a stationary EPW is given in the wave frame by $\dot{W}(t) = m\upsilon(t)a(t)$ and in the plasma frame by $\dot{W}(t) = m[\upsilon(t) - u]a(t)$. In both cases the final velocity $\upsilon(t)$ and acceleration *a(t)* are given in terms of initial conditions (x',υ') in the wave frame so that in both cases υ' will be distributed according to $\phi_o(\upsilon'-u)$. In the wave frame we have

$$n_o\langle\dot{W}(t)\rangle_{x'}=n_o\frac{m}{l}\int_0^l\left[\upsilon'+\varepsilon\dot{b}_1(t)+\varepsilon^2\dot{b}_2(t)+\cdots\right]\left[\varepsilon\ddot{b}_1(t)+\varepsilon^2\ddot{b}_2(t)+\cdots\right]dx' \tag{7}$$

$$n_o\langle\dot{W}(t)\rangle_{x',\upsilon'}=m\int_{-\infty}^{+\infty}\left\{\varepsilon^2\left[\upsilon'\ddot{T}_2(\upsilon',t)+R_2(\upsilon',t)\right]+\varepsilon^4\left[\upsilon'\ddot{T}_4(\upsilon',t)+R_4(\upsilon',t)\right]\right\}\phi_o(\upsilon'-u)d\upsilon' \tag{8}$$

It can be shown [9] that for $t>>t_o$ we have

$$\ddot{T}_2(\upsilon',t)\approx\frac{\pi}{2k}\frac{\partial\delta(\upsilon')}{\partial\upsilon'};\ \ \ddot{T}_4(\upsilon',t)\approx-\frac{\pi}{384}kt^4\frac{\partial\delta(\upsilon')}{\partial\upsilon'}$$

$$R_2(\upsilon',t)\approx\frac{\pi}{2k}\delta(\upsilon')\ ;\ \ R_4(\upsilon',t)\approx-\frac{\pi}{384}kt^4\delta(\upsilon') \tag{9}$$

Substituting Eqs.(9); into Eq.(8) we find in the wave frame:

$$n_o\langle\dot{W}(t)\rangle_{x',\upsilon'}=\frac{\pi}{2}n_o\frac{m\varepsilon^2}{k}\left\{1-\frac{1}{192}\varepsilon^2k^2t^4\right\}\int_{-\infty}^{+\infty}\left\{\upsilon'\frac{\partial\delta(\upsilon')}{\partial\upsilon'}+\delta(\upsilon')\right\}\phi_o(\upsilon'-u)d\upsilon'=0 \tag{10}$$

Thus in the plasma frame the energy rate simplifies as follows

$$n_o\langle\dot{W}(t)\rangle_{x',\upsilon'}=-n_om\langle\left[\upsilon(t)-u\right]a(t)\rangle_{x',\upsilon'}=-un_o\langle F(t)\rangle_{x',\upsilon'} \tag{11}$$

where $\langle F(t)\rangle_{x',\upsilon'}$ was derived for $t>>t_o$ up to fourth order in amplitude in Eq.(6) so that in the plasma frame the energy rate is given by

$$n_o\langle\dot{W}(t)\rangle_{x',\upsilon'}\approx-\frac{\pi}{2}n_o\frac{m\varepsilon^2}{k}\frac{\omega}{k}\left[\frac{\partial\phi_o(\upsilon)}{\partial\upsilon}\right]_{\upsilon=\omega/k}\left\{1-\frac{1}{192}k^2\varepsilon^2t^4+\cdots\right\} \tag{12}$$

As in he case of the force [Eq.(6)], the nonlinear evolution of $n_o\langle\dot{W}(t)\rangle_{x',\upsilon'}$ is governed during transition from the linear to the nonlinear regime by the *universal function* $M(\tau)\approx1-(\tau^4/192)+O(\tau^8)$ where $\tau=t/t_{osc}$. Note that if the wave amplitude is slowly varying and the gain of energy of the plasma is equated to the loss of energy of the wave, it can be easily proved from Eq.(12) that $M(\tau)=\gamma(\tau)/\gamma_L$ where $\gamma(\tau)$, γ_L are the Landau and O'Neil damping rates respectively.

An approximate formula for the long time behaviour of the force exerted on a plasma by a stationary EPW can be obtained by expanding Eqn.(3) within the argument:

$$F(t)=m\varepsilon\cos\{k[x'+\upsilon't+\varepsilon b_1(t)]\}=m\varepsilon\cos\left[kx'+k\upsilon't+\mu\sin(kx'+\psi)\right] \tag{13}$$

where ψ,μ are independent of x'. Taking the space average of $F(t)$ and using $\theta=kx'+\psi$ we obtain

$$\langle F(t)\rangle_{x'} = \frac{m\varepsilon}{2\pi}\int_0^{2\pi}\cos(\theta - \psi + kv't + \mu\sin\theta)d\theta = 2m\varepsilon^2\frac{J_1(\mu)}{\mu}\ddot{T}_2(v',t) \tag{14}$$

where $J_1(\mu)$ is the *Bessel function* and μ is given explicitly as

$$\mu = \frac{\varepsilon}{kv'^2}\left\{ [1-\cos(kv't)]^2 + [kv't - \sin(kv't)]^2 \right\}^{1/2} \tag{15}$$

For $t >> t_o$ we have $\ddot{T}_2(v',t) \approx (\pi/2k)\partial\delta(v')/\partial v'$ so that

$$n_o\langle F(t)\rangle_{x',v'} = \pi n_o\frac{m\varepsilon^2}{k}\int_{-\infty}^{+\infty}\frac{J_1(\mu)}{\mu}\frac{\partial\delta(v')}{\partial v'}\phi_o(v'-u)dv'$$

$$= -\frac{\pi}{2}n_o\frac{m\varepsilon^2}{k}\left[\frac{\partial\phi_o(v)}{\partial v}\right]_{v=\omega/k}\left\{\frac{4J_1(\varepsilon kt^2/2)}{\varepsilon kt^2}\right\} \tag{16}$$

We observe that the above result is a product of the linear damping force and a *universal function* $M(\tau)$ which in the present approximation has the form $M(\tau) \approx (4/\tau^2)J_1(\tau^2/2)$; $\tau = t/t_{osc}$ characterized for $t >> t_{osc}$ by an oscillatory decay.

In conclusion, the wavelet representation of the solution of the equations of motion provides a new framework for the evaluation of macroscopic variables in the theory of wave-plasma interactions and in particular the nonlinear problem is reduced to the determination of a *universal function* $M(\tau)$ which is independent of the velocity distribution.

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