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Role of trapped particles in linear wave-plasma interactions

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When a stationary (constant amplitude) electrostatic wave is running through a collisionless plasma, the plasma particles can be separated into *trapped* and *untrapped* according to their initial positions and velocities with respect to the wave. The question which of the above two groups is responsible for the linear wave-plasma interactions leading to Landau damping [1] has been first considered by Bohm and Gross [2] and Jackson [3] who have shown by a semi-quantitative analysis that trapped particles imply linear damping or growing of plasma waves. In parallel Dawson [4] has shown that a stationary electrostatic wave interacts with both trapped and untrapped particles of the plasma if the velocities of these particles are near the phase velocity of the wave (resonant particles). Later Stix [5] obtained the same result by considering the wave-plasma interaction as an initial value problem using particle dynamics. No separation between groups of particles was made in this work but it became apparent that *resonant* particles become gradually important as $t >> t_o$; $t_o = (m/k^2T)^{1/2}$ (phase mixing time). On the other hand, the nonlinear theory of plasma oscillations developed later by O'Neil [6] has brought again the trapped particles into the picture since it was shown that nonlinear wave-plasma interactions are due separately to trapped and untrapped particles and decay in time as function of t/t_{osc} where $t_{osc} = (m/kqE_o)^{1/2}$ is the trapped particle oscillation period. Thus, although it is now accepted [7] that trapped particle oscillations in the trough of the wave are responsible for the nonlinear saturation of plasma waves due to the decay of wave-plasma interactions, the precise contribution of trapped particles to linear Landau damping is still unclear.

In the present article we first calculate the number density of trapped particles of a Maxwellian plasma under the influence of a stationary electrostatic wave in terms of the amplitude and the phase velocity of the wave. Then, by averaging the amplitude expansion [8] of the motion of each plasma particle up to second order (linear theory) over initial conditions, we derive the total force exerted by the wave on the plasma particles (trapped+untrapped) and the particular force exerted by the wave on the trapped particles only. By comparing these two forces for various time scales, the role of the trapped particles in the linear wave-plasma interactions is clarified.

Number density of trapped particles

A plasma particle is trapped in the trough of a stationary electrostatic wave $E = E_o \cos(kx - \omega t)$ if its initial conditions (x', υ') satisfy

$$-\Delta\upsilon < \upsilon' < \Delta\upsilon\,;\ \Delta\upsilon \equiv \left(2\varepsilon / k\right)^{1/2} \sqrt{1+\sin(kx')} \tag{1}$$

where $\varepsilon = qE_o / m$. The number density of trapped particles at x' for a Maxwellian plasma is given by

$$n_{tr}(x') = n_o \left(m / 2\pi T\right)^{1/2} \int_{-\Delta\upsilon}^{+\Delta\upsilon} e^{-\frac{m}{2T}(\upsilon' - u)^2} d\upsilon' \tag{2}$$

where $u = -\omega / k$ is the fluid velocity of the plasma with respect to the wave. Splitting the integral into two parts we have

$$n_{tr}(x') = \frac{1}{2} n_o \left\{ erf\left[\varepsilon^{*1/2}\sqrt{1+\sin(kx')} + u^* / \sqrt{2}\right] + erf\left[\varepsilon^{*1/2}\sqrt{1+\sin(kx')} - u^* / \sqrt{2}\right]\right\} \tag{3}$$

where $\varepsilon^* = qE_o / Tk$ and $u^* = u(m/T)^{1/2}$. Taking the average of $n_{tr}(x')$ over a wavelength $l = 2\pi k$ and introducing $\theta = kx'$ we have

$$\langle n_{tr} \rangle = \frac{1}{l}\int_0^l n_{tr}(x')dx' = \frac{n_o}{4\pi}\int_0^{2\pi} \left[erf(u^* + \alpha) - erf(u^* - \alpha)\right] d\theta \tag{4}$$

where $\alpha = \varepsilon^{*1/2}\sqrt{1+\sin\theta}$. Expanding the error functions at $\alpha = 0$ (small amplitude limit) and integrating order by order we have:

$$\langle n_{tr} \rangle = \frac{4\sqrt{2}}{\pi^{3/2}} n_o \varepsilon^{*1/2} e^{-u^{*2}} \left\{1 + \frac{4(2u^{*2} - 1)}{9}\varepsilon^* + \cdots\right\} \tag{5}$$

We notice that $\varepsilon^* = (t_o / t_{osc})^2$ so that the above approximation is valid only if $t_o << t_{osc}$. As expected the average number density of trapped particles decreases gradually as the amplitude E_o goes to zero, however since the rate of decrease is slow ($\sim E_o^{1/2}$), trapped particles are not negligible for small amplitude plasma waves.

Contribution of trapped particles to linear wave-plasma interactions

In the wave frame the equations of motion for individual plasma particles in the presence of a stationary electrostatic wave read

$$\frac{dx}{dt} = \upsilon\ ;\quad \frac{d\upsilon}{dt} = \varepsilon\cos(kx) \tag{6}$$

Expanding the solution of Eqs. (6) in powers of ε the force exerted on a plasma particle by the wave can be expressed as follows

$$F(t) = m\left[\varepsilon\ddot{b}_1(t) + \varepsilon^2\ddot{b}_2(t) + \varepsilon^3\ddot{b}_3(t) + \cdots\right] \tag{7}$$

where $\ddot{b}_1(t), \ddot{b}_2(t)$ have the form

$$\ddot{b}_1(t) = \ddot{P}_{11}(\upsilon',t)\cos(kx') + \ddot{Q}_{11}(\upsilon',t)\sin(kx')$$

$$\ddot{b}_2(t) = \ddot{T}_2(\upsilon',t) + \ddot{P}_{22}(\upsilon',t)\cos(2kx') + \ddot{Q}_{22}(\upsilon',t)\sin(2kx')$$

Averaging (7) over a wavelength $l = 2\pi k$ and a Maxwellian distribution translated by $u = -\omega / k$ in the wave frame, the *total* force per unit volume exerted on the plasma particles by the wave up to second order in amplitude (linear force) reads

$$n_o\langle F(t)\rangle_{x',\upsilon'} = n_o m\varepsilon^2\left(m/2\pi T\right)^{1/2}\int_{-\infty}^{+\infty}\ddot{T}_2(\upsilon',t)e^{-\frac{m}{2T}(\upsilon'-u)^2}d\upsilon' \tag{8}$$

where the wavelet $\ddot{T}_2(\upsilon',t)$ is given by

$$\ddot{T}_2(\upsilon',t) = \frac{1}{2k\upsilon'^2}\left[k\upsilon't\cos(k\upsilon't) - \sin(k\upsilon't)\right] \tag{9}$$

For short times $t << t_o$ we have $\ddot{T}_2(\upsilon',t) \approx -(1/6)k^2\upsilon't^3$ so that the force becomes

$$n_o\langle F(t)\rangle_{x',\upsilon'} = -\frac{1}{6}n_o m\varepsilon^2 k^2 t^3 u \tag{10}$$

For long times $t >> t_o$ we have $\ddot{T}_2(\upsilon',t) \approx (\pi/2k)\partial\delta(\upsilon')/\partial\upsilon'$ and the force reaches the Landau plateau:

$$n_o\langle F(t)\rangle_{x',\upsilon'} = -\sqrt{\frac{\pi}{8}}n_o q^2 E_o^2\frac{m^{1/2}}{kT^{3/2}}ue^{-\frac{mu^2}{2T}} \tag{11}$$

The above force characterises the linear wave-plasma interactions for both trapped and untrapped particles and acts in the direction of propagation of the electrostatic wave in the plasma.

The force per unit volume exerted *only* on the trapped particles of the plasma by the wave is given by

$$n_o\langle F_{tr}(t)\rangle_{x',\upsilon'} = \frac{n_o}{2\pi}m\varepsilon^2(m/2\pi T)^{1/2}\int_0^{2\pi}d\theta\int_{-\Delta\upsilon}^{+\Delta\upsilon}\ddot{b}_2(t)e^{-\frac{m}{2T}(\upsilon'-u)^2}d\upsilon'$$

$$= \frac{n_o}{\pi k}m\varepsilon^2(m/2\pi T)^{1/2}e^{-\frac{mu^2}{2T}}\left\{\int_0^{2\pi}d\theta\sin^2\theta\int_0^{\Delta\upsilon}\frac{k\upsilon't - \sin(k\upsilon't)}{\upsilon'^2}\cos(k\upsilon't)e^{-\frac{m\upsilon'^2}{2T}}\sinh\left(\frac{mu}{T}\upsilon'\right)d\upsilon'\right.$$

$$\left. - \int_0^{2\pi}d\theta\cos^2\theta\int_0^{\Delta\upsilon}\frac{1-\cos(k\upsilon't)}{\upsilon'^2}\sin(k\upsilon't)e^{-\frac{m\upsilon'^2}{2T}}\sinh\left(\frac{mu}{T}\upsilon'\right)d\upsilon'\right\} \tag{12}$$

For short times $t << t_o << t_{osc}$, and intermediate times $t_o << t << t_{osc}$ we obtain

$$n_o \langle F_{tr}(t) \rangle_{x',v'} = -\frac{1}{8} n_o m \varepsilon^2 k^2 (t_o / t_{osc})^3 u e^{-\frac{mu^2}{2T}} t^3 \tag{13}$$

Comparing the above result with the total force given by Eq.(10) we have

$$R \equiv n_o \langle F_{tr}(t) \rangle_{x',v'} / n_o \langle F(t) \rangle_{x',v'} = 0.75 (t_o / t_{osc})^3 \exp(-mu^2 / 2T) \tag{14}$$

We observe that for $t << t_{osc}$ the trapped particles make a very small contribution to the wave-plasma interactions $(R << 1)$.

For long times $t_o << t_{osc} << t$ Eqn.(12) becomes

$$n_o \langle F_{tr}(t) \rangle_{x',v'} = \frac{n_o}{\sqrt{2\pi}} q^2 E_o^2 \frac{m^{1/2}}{kT^{3/2}} u e^{-\frac{mu^2}{2T}} \left\{ \int_0^\infty \cos z e^{-cz^2} dz - \int_0^\infty \frac{\sin z}{z} e^{-cz^2} dz \right\} \tag{15}$$

where $c = t_o^2 / 2t^2$. At the limit $c \to 0$ $(t_o << t)$ the above force coincides with the plateau of the total force (11). Therefore it is clear that for long times $t >> t_{osc}$ the total linear force is due to trapped particles only.

In conclusion, it was demonstrated that for a plasma where $t_o << t_{osc}$ $(qE_o / kT << 1)$ the linear wave-plasma interactions are mainly due to untrapped particles for $t << t_{osc}$ and to trapped particles for $t >> t_{osc}$, although in the second case the linear wave-plasma interactions should be corrected due to nonlinear effects.

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