

## Exact periodic solutions of the Liouville equation and the “snake” of density in JET

F. Spineanu and M. Vlad

*Association Euratom-MEC Romania*

*NILPRP, P.O.Box MG-36, Magurele, Bucharest, Romania*

The *snake* phenomenon has been observed in JET during the experiments of pellet injection and consists in formation of persistent density perturbations at rational- $q$  surfaces. These structures persist over several sawtooth collapses and are difficult to explain as magnetic perturbations. Possibly related to this, there are indications that the tokamak plasma density has an anomalous radial pinch, much larger than that of the neoclassical origin. In this class of phenomena one should also include the persistent impurity accumulation in laser blow-off injected impurity, observed in experiments in TCV. There are several studies of the statistical properties of the correlations between the peaking factors for density, current density or pressure, with plasma parameters and these studies seem to support the idea of turbulent equipartition of the thermodynamic invariants (Minardi). However we should note that these studies involve quantities expressed as global variables (like averages) and they can hide other dependences not immediately obvious.

We consider the possibility that the particle density behavior (and particularly the *snake* phenomenon) can be connected with the existence of attracting solutions of certain nonlinear integrable equations. The statistical studies carried out on a large set of discharges with the purpose of testing the prediction of the Turbulent Equipartition theory have suggested that the current density is given by the equation  $\Delta j + (\lambda^2/4)j = 0$  where  $\lambda$  is a constant. However this may be valid on finite spatial *patches* and on every patch the equation can be considered an approximation of some more general equation, for which the derivation from first principles may be possible. One is the *sinh*-Poisson equation,  $\Delta\phi + (\lambda^2/4)\sinh\phi = 0$  for which the previous equation is the linearised form. The reason to consider the *sinh*-Poisson equation comes from the existing proofs that this equation governs the asymptotic states of ideal fluids, or, more generally, of  $2D$  systems that can be reduced to the dynamics of point-like elements interacting by the potential which is the inverse of the Laplacean operator (Jackiw and Pi, Spineanu and Vlad). This equation is however obtained when there are two kinds of elements (like positive and negative vorticity) and they are of equal number,  $n_+ = n_-$ . Then the *sinh*-Poisson equation is obtained as governing the states with maximum entropy of the discrete statistical system at negative temperature (Montgomery *et al.*). Since the equation for the current density mentioned above is derived under the assumption of turbulent equipartition, the two descriptions may be related. However, the solutions for the *unbalanced* system of elements,  $\alpha \equiv n_+/n_- \neq 1$ ,  $\Delta\phi = (\lambda^2/8)[\exp(\phi)/\sqrt{\alpha} - \sqrt{\alpha}\exp(-\phi)]$  have been obtained numerically (Pointin and Lundgren) and have been shown to have higher entropy and higher stability than those of the *sinh*-Poisson, precisely the characteristics we are seeking for. The limiting form of the unbalanced equation is the Liouville equation,  $\Delta\phi = (\lambda^2/8)\exp(-\phi)$ .

There are field theoretical models that are able to describe statistical ensembles of a discrete sets of elements and these models lead to the Liouville equation when a particular condition (called : *self-duality*) is fulfilled. We note that the Liouville equation can lead to the above current density equation (Helmholtz-type) in an approximation where the function differs weakly from a background value.

In the following we will discuss solutions of the soliton type of the Liouville equation  $\Delta\phi = (\lambda^2/8) \exp(-\phi)$ . It is commonly considered that the Liouville equation is solved by the formula given in terms of two arbitrary complex functions  $F$  and  $G$

$$\phi(x, y) = -\ln \left\{ \frac{F'(z) G'(z^*)}{\left[1 + \frac{\lambda^2}{16} F(z) G(z^*)\right]^2} \right\}$$

where  $z = x + iy$  and  $*$  is the complex conjugate. However this form of the solution is too general to be useful. More popular is the particular form derived from this one,

$$\phi = \ln [\cosh(kx) + \varepsilon \cos(ky)]$$

solution of  $\Delta\phi = (1 - \varepsilon^2) k^2 \exp(-2\phi)$ . This describes the *cat's eye* vortex chains of the Kelvin-Stuart stream function, and the magnetic flux function in the chains of magnetic islands (Finn and Kaw).

Our main objective is to identify a type of solution of the Liouville equation which is more localised on a magnetic surface compared with the profile of a magnetic island of the same helical symmetry. For this the factor  $k$  must be independent of the local  $q$ : the spatial extension of the density perturbation at a magnetic surface and the helical symmetry of that magnetic surface must be independent. This is clear in the case of the Liouville (and *sinh*-Poisson) equation, since it is *conformally* invariant. We cannot be sure that the solution we are looking for can be of the form shown above. The correct approach is to find any solution of the Liouville equation.

We now describe the systematic way of obtaining solutions of the Liouville equation on periodic domains, taking  $y$  as the poloidal and  $x$  as the radial coordinate. The method (Tracy *et al.*) consists of taking the Liouville equation as the limit of the *sinh*-Poisson equation:

$$\lambda^2 = \tilde{\lambda}^2 \exp(-\beta), \quad \psi \equiv \phi - \beta$$

and let  $\beta \rightarrow \infty$ . Then the *sinh*-Poisson equation becomes the Liouville equation after taking  $\tilde{\lambda} \rightarrow \lambda$ . The *sinh*-Poisson equation is exactly integrable on periodic domains since it possesses a pair of Lax operators (Ting, Chen and Lee). The eigenvalue problem for the Lax operators identifies a spectrum of complex values where the two Bloch functions are not independent (for one eigenvalue we have only one eigenfunction). These nondegenerate eigenvalues are called *main spectrum* and represent branching points in the two-sheeted Riemann surface associated to the Wronskian of the two periodic solutions. It is shown that the problem of finding the unknown function  $\phi$  is mapped on the problem of motion of a set of functions (*auxiliary spectrum*) on this Riemann surface. The equations of motion are nonlinear but they can be solved exactly, using the Abel transform. In this transformation the two-sheeted Riemann surface is mapped onto a compact hyperelliptic Riemann surface and the equations of motion are linearised as rotations along the cycles on this complex curve. The topology of the hyperelliptic Riemann surface is that of a sphere with a number of handles (the *genus* of the surface) which is given by the geometry of the cuts in the complex plane needed to uniformize the first Riemann surface. Therefore the *genus* is actually determined by the *main spectrum* of the Lax linear operator, or, in concrete terms, by the boundary condition we require for the solution. The linear equations of motion are integrated leading to *phases*, which are linear combinations of the original variables,  $x$  and  $y$ . The number of phases is the number of type *A* cycles on the surface (like the short turn on a torus), or, the *genus* ;

so, the number of phases is again determined by the *main spectrum*. We have to return to the original framework, and this represents the Jacobi inversion problem. It is solved exactly in terms of Riemann *theta* functions. The exact solution of the *sinh*-Poisson equation in terms of Riemann *theta* function,  $\Theta$ , is

$$\phi(x, y) = 2 \ln \left( \frac{\Theta(\mathbf{l} + \frac{1}{2}\mathbf{1})}{\Theta(\mathbf{l})} \right)$$

where  $\mathbf{l} = \mathbf{k}_x x + \mathbf{k}_y y + \mathbf{l}_0$ ,  $\mathbf{l}_0$  is a vector of constants, initial phases, and

$$\begin{aligned} k_{x,j} &\equiv (-1)^N \frac{C_{jN}}{8\sqrt{Q}} + 2C_{j1} \\ k_{y,j} &\equiv i(-1)^N \frac{C_{jN}}{8\sqrt{Q}} - 2iC_{j1} \end{aligned}$$

The physical content of the problem is in the square matrix  $C$  whose dimension is half the number of eigenvalues in the main spectrum. The matrix  $C$  is obtained from integrals of a basis of differential one-forms defined on the hyperelliptic Riemann surface along the basis of closed paths (cycles). These integrals can be converted into integrals along closed paths on the plane of the spectral variable, around cuts or crossing these cuts. The geometrical aspect of this conversion is numerically complicated due to the jumps of the phases of the complex integrand at crossing the cuts. However, the symmetries of the main spectrum allows us to use general forms of the matrix

$$\begin{aligned} C_{ij} &= 16^{N-2j+1} C_{i,N-j+1}^*, \quad j \leq N/2 \\ C_{ij} &= C_{N-i+1,j}^* \end{aligned}$$

A particular choice of the entries of  $C$  (which obeys the symmetries) corresponds,

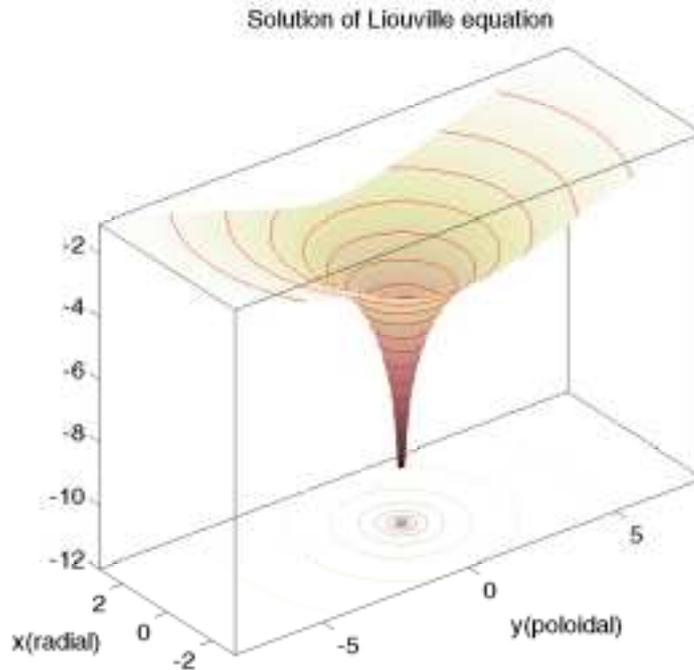


Figure 1: The streamfunction  $\phi$  solution of the Liouville equation.

physically, to a particular form of the boundary condition assumed for  $\phi$ , on a linear section of the periodic domain.

The solution of the Liouville equation can be obtained from that for the *sinh*-Poisson equation in a certain limit. This limit has been translated into a particular distribution of the functions of the auxiliary spectrum (Tracy *et al.*). For any  $(x, y)$  there are three classes according to the positions relative to the inversion circle, which is given by  $|E|^2 = \lambda^4/256$ ,  $E$  being the spectral variable. First, one notes that the discrete points of the *main spectrum* are situated in certain positions around this circle: (1) there are  $N$  inversion pairs,  $(E_j, E_{N+j})$ , in the set  $E_1, E_2, \dots, E_{2N}$  with:  $E_{N+j} = \lambda^4/(256E_j^*)$  for  $j = 1, N$ . (2) there are  $M$  pairs  $E_{2N+1}, \dots, E_{2N+2M}$  such that  $E_{2N+k} = \lambda^2\alpha_k/16$ ,  $E_{2N+M+k} = (\lambda^2/16)/\alpha_k^*$ , with  $\alpha_k$  independent of  $\lambda$ . From each pair of the eigenvalues of the set (2) (situated near the inversion circle) there are  $M$  auxiliary functions, which scale as  $\lambda^2$ . The rest of the points of the auxiliary spectrum are divided into two classes. The first contains the auxiliary functions which are outside the inversion circle and are independent of  $\lambda$ . The second are defined inside the inversion circle and are scaled as  $\lambda^4$ . In this way,  $\lambda^2 \exp(-\phi)$  is independent of  $\lambda$ . This classifications cannot be directly employed, but they suggest a particular choice for the points of the main spectrum. These studies, mainly numerical, are still in progress.

A conclusion can be drawn at this stage: the solution exhibits a localised perturbation on the poloidal direction, while the helical symmetry is still that given of the  $q$  of the surface. This solution is solitonic and therefore is stable (we still have to clarify the effect of the limiting procedure on the set of invariants when passing from *sinh*-Poisson to Liouville). The fact that this kind of solution is an attractor comes from the general property of the plasma state: the self-duality (leading to the Liouville equation) corresponds to the extremum of the action of the system of discrete elements.

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