31st EPS Conference on Plasma Phys. London, 28 June - 2 July 2004 ECA Vol.**28G**, P-4.204 (2004)

Theoretical studies of particle diffusion due to fluctuations

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The anomalous transport caused by electrostatic fluctuations is studied in a strong magnetic field by using a renormalization technique of statistical dynamics. With the use of the functional integral method [1,2], a closed set of equations for the ensemble-averaged distribution function and the response function to an infinitesimal external perturbation is derived within the framework of the direct-interaction approximation (DIA). From these equations, we first discuss the cross-field particle diffusion coefficient and the anomalous diffusion of the current density driven by the radio-frequency (rf) waves. We next extend the DIA by using the idea of subensemble average [3], and incorporate the effect of particle trajectory into our functional integral formulation.

We start with a kinetic equation for an electron distribution function f in a presence of low-frequency electrostatic fluctuations

$$\partial_t f + \delta \boldsymbol{v}_\perp \cdot \nabla_\perp f - C(f) = S, \tag{1}$$

where C is the collision operator; the velocity fluctuation due to fluctuating electric field $\delta \boldsymbol{E}_\perp$ is written as $\delta \boldsymbol{v}_\perp = \delta \boldsymbol{E}_\perp \times \boldsymbol{B}/B^2$ with an averaged magnetic field $\boldsymbol{B}$; and S represents a source term. Applying the functional integral method to this kinetic equation, we will derive a closed set of equations for ensemble-averaged quantities. Let us now write the distribution function as $f = \psi(\boldsymbol{r}, \boldsymbol{p}, t) f_0(p)$, and introduce a generating functional

$$Z[\zeta, \xi] = \mathcal{N} \int \mathcal{D}[\psi']\mathcal{D}[\hat{\psi}] \;\; e^{-\mathcal{S}} \equiv e^W \tag{2}$$

with

$$\mathcal{S} = \hat{\psi}(1) \left[\partial_t \psi'(1) - \hat{C}(\psi'(1)) - \hat{S}\right] - \psi'(1)\eta(1) - \hat{\psi}(1)\xi(1) - \ln\langle e^{-\hat{\psi}(1)\delta \boldsymbol{v}_\perp \cdot \nabla_\perp \psi'(1)}\rangle, \tag{3}$$

where $1 = (\boldsymbol{r}, \boldsymbol{p}, t)$; $\hat{C}(\psi') = C(\psi' f_0)/f_0$ and $\hat{S} = S/f_0$; f_0 is the Maxwellian with the constant number density $\bar{n}$ and the constant temperature; $\langle \cdot \rangle$ denotes the ensemble average over fluctuations; $\int \mathcal{D}[\psi']\mathcal{D}[\hat{\psi}]$ means the functional integral; $\mathcal{N}$ is the multiplicative constant irrelevant to the following calculation; and the integration over repeated indices is assumed in the case of no confusion. The generating functional (2) yields evolution equations for $g(1) \equiv \delta W/\delta\eta(1)$ and $G(1,1') \equiv \delta^2 W/\delta\xi(1')\delta\eta(1)$, where the one-point function $g(1)$ becomes the ensemble-averaged distribution function in the limit of $\eta = \xi = 0$, i.e. $g(1)|_{\eta=\xi=0} =< \psi > /f_0$. Performing the ensemble average in (3) with Gaussian assumption for $\delta \boldsymbol{v}_\perp$, and neglecting the three point function $\delta G(1,2)/\delta\xi(1')$, we can write the evolution equations for g and G in the form:

$$\left(\partial_t - \hat{C}\right) g(1) - \nabla_\perp \cdot \boldsymbol{F}(\mathbf{1}, \mathbf{1'}) G(1,1') \cdot \nabla'_\perp g(1') = \hat{S}, \tag{4}$$

$$\left(\partial_t - \hat{C}\right) G(1, 1'') - \nabla_\perp \cdot \boldsymbol{F}(\mathbf{1}, \mathbf{1}') G(1, 1') \cdot \nabla'_\perp G(1', 1'') = \delta(1 - 1''), \qquad (5)$$

where we assume the form of the correlation function as $\boldsymbol{F}(\mathbf{1}, \mathbf{1}') = \langle \delta \boldsymbol{v}_\perp(\boldsymbol{r}_\perp) \delta \boldsymbol{v}_\perp(\boldsymbol{r}'_\perp) \rangle \equiv \boldsymbol{F}(\boldsymbol{r}_\perp - \boldsymbol{r}'_\perp)$. The approximation leading to the closed set of equations for g and G is referred to as the DIA.

We here define the cross-field diffusion tensor in terms of the ensemble-averaged distribution function as

$$\boldsymbol{D}_\perp = \frac{1}{2N} \frac{d}{dt} \int d\boldsymbol{r} d\boldsymbol{p}\, \boldsymbol{r}_\perp \boldsymbol{r}_\perp g(\boldsymbol{r}, \boldsymbol{p}, t) f_0(p) = \frac{1}{2N} \int d\boldsymbol{r} d\boldsymbol{p}\, \boldsymbol{r}_\perp \boldsymbol{r}_\perp \frac{\partial g}{\partial t} f_0(p) \qquad (6)$$

with the particle number N . The anomalous part of this diffusion tensor is obtained by substituting the equation (4) into (6) as

$$\boldsymbol{D}^{\mathrm{an}}_\perp = \frac{1}{\bar{n}} \int d\boldsymbol{p} f_0(p) \boldsymbol{D}_p(p), \quad \boldsymbol{D}_p(p) = \int d\boldsymbol{r}'_\perp d\boldsymbol{p}' \boldsymbol{F}(\boldsymbol{r}_\perp - \boldsymbol{r}'_\perp) G^\dagger(\boldsymbol{r}_\perp - \boldsymbol{r}'_\perp, \boldsymbol{p}, \boldsymbol{p}'), \quad (7)$$

where we have used the approximation $g(\boldsymbol{r}, \boldsymbol{p}, t) \simeq n(\boldsymbol{r})/\bar{n}$ with the number density $n(\boldsymbol{r})$. The response function $G^\dagger$ in (7) is defined by

$$G^\dagger(\boldsymbol{r}_\perp - \boldsymbol{r}'_\perp, \boldsymbol{p}, \boldsymbol{p}') = \int dt' \int dz' G(t - t', \boldsymbol{r}' - \boldsymbol{r}, \boldsymbol{p}', \boldsymbol{p}) \frac{f_0(p')}{f_0(p)}, \qquad (8)$$

and this function is determined by the equation

$$-\hat{C}(G^\dagger(\boldsymbol{r}_\perp - \boldsymbol{r}''_\perp, \boldsymbol{p}, \boldsymbol{p}'')) - \nabla_\perp \cdot \int d\boldsymbol{r}'_\perp d\boldsymbol{p}' \boldsymbol{F}(\boldsymbol{r}_\perp - \boldsymbol{r}'_\perp) G^\dagger(\boldsymbol{r}_\perp - \boldsymbol{r}'_\perp, \boldsymbol{p}, \boldsymbol{p}')$$

$$\cdot \nabla'_\perp G^\dagger(\boldsymbol{r}'_\perp - \boldsymbol{r}''_\perp, \boldsymbol{p}', \boldsymbol{p}'') = \delta(\boldsymbol{r}_\perp - \boldsymbol{r}''_\perp) \delta(\boldsymbol{p} - \boldsymbol{p}''). \qquad (9)$$

In order to obtain the response function $G^\dagger$, we need to solve the rather complicated integro-differential equation (9). This complicated equation is quite simplified by using the local (Markovian) approximation in configuration and momentum spaces. Using the high-energy form of the collision operator

$$\hat{C} = \nu_c m c^3 \left[-\frac{1}{v^2} \frac{\partial}{\partial p} + \frac{(1+Z)m}{vp^2} \frac{1}{2} \frac{\partial}{\partial \zeta} (1 - \zeta^2) \frac{\partial}{\partial \zeta} \right] \qquad (10)$$

and solving the equation (9) with the use of local approximation, we can obtain the following expression for $\boldsymbol{D}_p$

$$\boldsymbol{D}_p(p) = \int \frac{d\boldsymbol{k}_\perp}{(2\pi)^2} \frac{\boldsymbol{F}(\boldsymbol{k}_\perp)}{\boldsymbol{k}_\perp \boldsymbol{k}_\perp : \boldsymbol{D}_p} \left\{ 1 - \exp\left[-\frac{1}{\nu_c} \boldsymbol{k}_\perp \boldsymbol{k}_\perp : \boldsymbol{D}_p(p) P(p) \right] \right\}, \qquad (11)$$

where $\zeta = p_\parallel / p$; $\nu_c = 4\pi \bar{n} e^4 \log \Lambda / m^2 c^3$; Z is the charge number; m and $-e$ are the mass and the charge of an electron; $P(p) = p/mc - \tan^{-1}(p/mc)$, and $\boldsymbol{F}(\boldsymbol{k}_\perp)$ is the Fourier transform of $\boldsymbol{F}(\boldsymbol{r}_\perp - \boldsymbol{r}'_\perp)$. Let assume that the correlation function of the electrostatic potential $\phi(\boldsymbol{r}_\perp)$ is gyrotropic (i.e. isotropic in the plain perpendicular to the averaged magnetic field) and it has the form of $\langle \phi(\boldsymbol{r}_\perp) \phi(\mathbf{0}) \rangle = \phi_0^2 \exp(-r_\perp^2 / 2\lambda_\perp^2)$. Then $\boldsymbol{D}^{\mathrm{an}}_\perp$ and $\boldsymbol{D}_p$ become diagonal tensors, and the diagonal component of $\boldsymbol{D}_p$ is obtained from (11) as $D_p(p) = \frac{\nu_c \lambda_\perp^2}{4P(p)}\left[\sqrt{1+2\left(\frac{2\beta P(p)}{\nu_c \lambda_\perp^2}\right)^2} - 1\right]$, where $\beta = \phi_0/B$. Substitution of this D_p into (7) yields the expression for the diagonal component of $\boldsymbol{D}_\perp^{\rm an}$.

We next consider the problem of the rf current drive in the presence of fluctuations. Applying the method discussed in ref 4. to the equation (4) and considering the momentum-space diffusion term $S = \partial \boldsymbol{p} \cdot \boldsymbol{S}_p$, we can derive the diffusion equation for the rf-driven current density J:

$$\nabla_\perp \cdot \boldsymbol{D}_{\rm rf} \cdot \nabla_\perp J - \nu_c J = -\nu_c J_0, \tag{12}$$

where J_0 is the rf-driven current density in the absence of fluctuations, and the spatial Markovian is assumed. The diffusion tensor for the current density is written as

$$\boldsymbol{D}_{\rm rf} = -\frac{\hat{\boldsymbol{s}} \cdot \frac{\partial}{\partial \boldsymbol{p}} \boldsymbol{\chi}_2}{\hat{\boldsymbol{s}} \cdot \frac{\partial}{\partial \boldsymbol{p}} \chi_1}, \tag{13}$$

where $\hat{\boldsymbol{s}} = \boldsymbol{S}_p / |\boldsymbol{S}_p|$; the function χ_1 that is a solution of the equation $\hat{C}(\chi_1) = \nu_c e v_\parallel$ is given by

$$\chi_1 = -ec\frac{p_\parallel}{p} G_1(p), \quad G_1(p) = \left(\frac{\gamma+1}{\gamma-1}\right)^{(Z+1)/2} \int_1^\gamma d\gamma' \left(\frac{\gamma'-1}{\gamma'+1}\right)^{(1+Z)/2} \frac{\gamma'^2 - 1}{\gamma'^2} \tag{14}$$

with $\gamma = \sqrt{1 + p^2/m^2c^2}$; and all the quantities are evaluated in the position of rf excitation in momentum space. The function $\boldsymbol{\chi}_2$ in (13) is obtained by solving the equation

$$\hat{C}(\boldsymbol{\chi}_2) = \nu_c \int d\boldsymbol{p}' d\boldsymbol{r}'_\perp \boldsymbol{F}(\boldsymbol{r}_\perp - \boldsymbol{r}'_\perp) \chi_1(\boldsymbol{p}') G^\dagger(\boldsymbol{r}'_\perp - \boldsymbol{r}_\perp, \boldsymbol{p}, \boldsymbol{p}'). \tag{15}$$

The diffusion tensor (13) becomes diagonal for the gyrotropic correlation function of the electrostatic potential. Then the diagonal components of $\boldsymbol{D}_{\rm rf}$ in the weak and strong turbulent limits are given by

$$\begin{aligned}\begin{pmatrix} D_{\rm rf}^{\rm weak} \\ D_{\rm rf}^{\rm strong} \end{pmatrix} &= \frac{1}{\hat{\boldsymbol{s}} \cdot \frac{\partial}{\partial \boldsymbol{p}}\left(\frac{p_\parallel}{p} G_1\right)} \hat{\boldsymbol{s}} \cdot \frac{\partial}{\partial \boldsymbol{p}} \frac{p_\parallel}{p} \left(\frac{\gamma+1}{\gamma-1}\right)^{(Z+1)/2} \\ &\times \int_1^\gamma d\gamma' \left(\frac{\gamma'-1}{\gamma'+1}\right)^{(Z+1)/2} \frac{\gamma'^2-1}{\gamma'^2} \begin{pmatrix} \frac{1}{2}\frac{\beta^2}{\nu_c \lambda_\perp^2}\left[P(p) - P(p')\right]^2 \\ \frac{\beta}{\sqrt{2}}\left[P(p) - P(p')\right] \end{pmatrix}. \end{aligned} \tag{16}$$

The DIA is the renormalized perturbation theory based upon the Eulerian picture. This Eulerian renormalization is known to fail in the strong turbulent limit due to the lack of proper treatment of particle trajectories. Recently, Vlad *et al.*[3] presented an interesting theory for estimating the running diffusion coefficient in the whole region from the quasi-linear to the strong turbulence. They have approximately taken into account the effect of particle trajectories in the fluctuating electrostatic potential by using the idea of subensemble average. We finally incorporate this idea of subensemble average into our functional integral formulation, and extend the DIA. Let us consider an ensemble average over fluctuating electric field under the conditions $\phi(\boldsymbol{r}_\perp, t) = \phi^0$ and $\delta\boldsymbol{v}_\perp(\boldsymbol{r}_\perp, t) = \boldsymbol{v}^0$, and write this subensemble average by $\langle\cdot\rangle_0$. In the similar way to (2), we introduce a following generating functional

$$Z_0[\eta, \xi] = \mathcal{N} \int \mathcal{D}[f']\mathcal{D}[\hat{f}] \;\; e^{-\mathcal{S}_0} \equiv e^{W_0} \tag{17}$$

with

$$\mathcal{S}_0 = \hat{f}(1)\partial_t f'(1) - f'(1)\eta(1) - \hat{f}(1)\xi(1) - \log\langle e^{-\hat{f}(1)\delta\boldsymbol{v}_\perp\cdot\nabla_\perp f'(1)}\rangle_0, \tag{18}$$

where $1 = (\boldsymbol{r}_\perp, t)$ and we have neglected the collisions for simplicity. We now define the subensemble-averaged response function $G_0(1-1') = \delta^2 W_0/\delta\xi(1')\delta\eta(1)$, and write the anomalous diffusion tensor in terms of this response function G_0 as

$$\boldsymbol{D}_\perp^{\mathrm{an}}(t) = \int d\phi^0 d\boldsymbol{v}^0 P_1(\phi^0)P_2(\boldsymbol{v}^0)\boldsymbol{D}_0(t), \quad \boldsymbol{D}_0(t) = \int_0^t d\tau d\boldsymbol{r}'_\perp \boldsymbol{E}(\boldsymbol{r}'_\perp, \tau)G_0(\boldsymbol{r}'_\perp, \tau), \tag{19}$$

where $\boldsymbol{E}(\boldsymbol{r}_\perp - \boldsymbol{r}'_\perp, t-t') = \langle\delta\boldsymbol{v}(\boldsymbol{r}'_\perp, t')\ \delta\boldsymbol{v}(\boldsymbol{r}_\perp, t)\rangle$, and $P_1(\phi^0)$ and $P_2(\boldsymbol{v}^0)$ are the Gaussian probability densities of ϕ^0 and $\boldsymbol{v}^0$. Repeating the procedure leading to (4) and (5), we can derive the equation for the response function $G_0(1-1')$

$$\begin{aligned}\frac{\partial G_0(1-1')}{\partial t'} \quad &+ \quad \frac{\phi^0}{\langle\phi^2(\boldsymbol{r}_\perp, t)\rangle}\boldsymbol{E}_\phi(1-1')\cdot\nabla' G_0(1-1') \\ &+ \quad \frac{2\boldsymbol{v}^0}{\langle\delta v^2(\boldsymbol{r}_\perp, t)\rangle}\cdot\boldsymbol{E}(1-1')\cdot\nabla' G_0(1-1') \\ &+ \quad \nabla'_\perp\cdot\boldsymbol{D}_0(t-t')\cdot\nabla'_\perp G_0(1-1') = -\delta(1-1'),\end{aligned} \tag{20}$$

where $\boldsymbol{E}_\phi(1-1') = \langle\delta\boldsymbol{v}(\boldsymbol{r}'_\perp, t')\phi(\boldsymbol{r}_\perp, t)\rangle$.

The solution to (20) in the approximation of neglecting $\boldsymbol{D}_0$ can be obtained by using the method of characteristics as $G_0(\boldsymbol{r}_\perp - \boldsymbol{r}'_\perp, t-t') = H(t-t')\delta(\boldsymbol{r}_\perp - \boldsymbol{r}'_\perp - \Delta\boldsymbol{X}_\perp(\boldsymbol{r}_\perp - \boldsymbol{r}'_\perp, t-t'))$, where $\Delta\boldsymbol{X}_\perp(\boldsymbol{r}_\perp - \boldsymbol{r}'_\perp, t-t')$ represents the perpendicular deviation of trajectory from the starting point $\boldsymbol{r}_\perp - \boldsymbol{r}'_\perp$ during the time $t-t'$. Then, with the approximation $\Delta\boldsymbol{X}_\perp(\boldsymbol{r}_\perp - \boldsymbol{r}'_\perp, t-t') \sim \Delta\boldsymbol{X}_\perp(\boldsymbol{0}, 0)$ in the calculation of diffusion tensor, we have

$$\boldsymbol{D}_0(t) = \int_0^t d\tau\boldsymbol{E}(\boldsymbol{X}_\perp(\tau), \tau), \tag{21}$$

where $\boldsymbol{X}_\perp(\tau)$ is determined by the equation

$$\frac{d\boldsymbol{X}_\perp(\tau)}{d\tau} = \boldsymbol{E}_\phi(\boldsymbol{X}_\perp(\tau), \tau)\frac{\phi^0}{\langle\phi^2(\boldsymbol{r}_\perp, t)\rangle} + 2\boldsymbol{E}(\boldsymbol{X}_\perp(\tau), \tau)\cdot\frac{\boldsymbol{v}^0}{\langle\delta v^2(\boldsymbol{r}_\perp, t)\rangle} \tag{22}$$

with the condition $\boldsymbol{X}_\perp(0) = \boldsymbol{0}$. This $\boldsymbol{X}_\perp(\tau)$ corresponds to the decorrelation trajectory proposed in ref. 3.

This work was partially supported by a Grant-in-Aid for Scientific Research (C) from Japan Society for the Promotion of Science.

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