*31st EPS Conference on Plasma Phys. London, 28 June - 2 July 2004 ECA Vol.**28G**, P-4.165 (2004)*

Bifurcational Solutions to Grad-Shafranov Equation

Pozdnyakov Yu.I

Russian Research Centre "Kurchatov Institute", Moscow, Russia

Spontaneous transitions from one steady state to another (so called L-H transition [1], internal transport barrier formation [2]), demonstrating typical bifurcational features, e.g., fast changes of equilibrium parameters (in particular, of an average plasma pressure), hysteresis, ets., are observed in tokamaks. There are many attempts to explain this phenomenon by bifurcation of transport coefficients, see, e.g., [3]. However, there are a few experimental evidences that the transitions do not have a local character but look rather like a global change of the plasma state. In particular, transport barriers may appear due to bifurcation of plasma equilibrium. This possibility was considered in [4]. The bifurcatonal mechanism itself was not discussed in that work. The idea that bifurcation of equilibrium can occur without breaking of the nested magnetic surfaces structure was recently proposed by E. Solano [5]. However, there were no calculations which would show bifurcational transitions explicitly. In the present paper we demonstrate the existence of multiple solutions to the classical axisymmetric equilibrium problem for plasma current column. A very simple current profile, which admits analytical solutions, is considered. The chosen parametrization of the problem allows us to reveal the bifurcational transition between these solutions. It is a nonlinearity of equilibrium equation that is responsible for the multiple solutions. Toroidal effects do not play an important role – the main result is obtained for a cylindrical column.

The axisymmetric plasma equillibrium is described in cylindrical coordinates r, φ, z by a well-known Grad-Shafranov equation [6],

$$\left(r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\right) + \frac{\partial^2}{\partial z^2}\right)\psi = -r^2 p' - FF', \tag{1}$$

which represents the projection of the vector force balance equation $\nabla p = \mathrm{rot}\mathbf{B} \times \mathbf{B}$ on the direction along $\nabla\psi$. Here ψ is a poloidal flux of the magnetic field $\mathbf{B}$, which can be written in a general form, $\mathbf{B} = [\nabla\psi \times \nabla\varphi] + F\nabla\varphi$. Two remaining components of the equation determine the functional dependence of plasma pressure p and poloidal current F: $p = p(\psi)$, $F = F(\psi)$. Consider a fixed-boundary problem. The function ψ defines the poloidal magnetic field flux to within the arbitrary constant, that's why the value of ψ on the boundary can be chosen zero, $\psi_b = 0$.

A well-known analytical method to solve equilibrium problem is the approxiamtion of almost circular magnetic surfaces [7]. In the main order of a/R, the equation for ψ can be written as

$$\frac{d}{da}\left(a\psi'(a)\right) = -aj(\psi), \tag{2}$$

which is equivalent to cylindrical approximation to within normalizing coefficients. The prime denotes the derivative with respect to the argument. In the right-hand-side of Eq.(2), the value of $j(\psi) = \frac{FF'_\psi}{R} + Rp'_\psi$ represents the "toroidal" current density (which is independent on r as it was expected in cylindrical approximation).

Here we normalized the variable a on the radius a_b of the cylinder, the current density $j(\psi)$ – on its characteristic value j_0, the flux ψ – on the value of $j_0 a_b^2 R$; the dimensionless values are denoted with the same letters.

In the fixed boundary problem, we may choose $\psi(1) = 0$ on the plasma boundary. The second boundary condition provides a regularity of the current in the column center: $\psi'(0) = 0$. Below we'll consider the graded current density profile (Fig.1),

$$j = 1 + j_1\theta(\psi - \psi_c), \tag{3}$$

where θ is Heavyside function, $j_1 = \text{const} \geq 0$. This choice is justified by the fact that on each segment with constant j equation (2) can be easily integrated. On the other hand, at finite j_1 the dependence $j(\psi)$ is essentially non-linear. Finally, the boundary problem in cylindrical approximation for $j(\psi)$ given by Eq.(3) reads as follows:

$$\frac{d}{da}\left(a\psi'(a)\right) = -a(1 + j_1\theta(\psi - \psi_c)), \quad \psi(1) = 0, \quad \psi'(0) = 0\ . \tag{4}$$

In this problem j_1, ψ_c are the parameters.

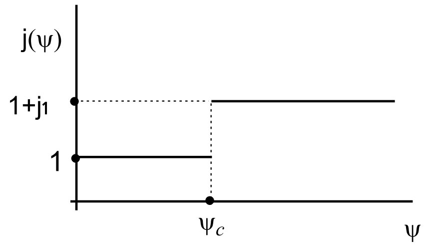

Fig. 1. Stepped current profile.

If there is no current jump ($j_1 = 0$), there is the only solution of the boundary problem (4), which is of parabolic type: $\psi(a) = (1 - a^2)/(4)$. If the current jump is positive ($j_1 > 0$) and $\psi_c > 1/4$, then the additional solutions appear,

$$\psi(a) = \begin{cases} \psi_c + \frac{\psi_0 - \psi_c}{j_1+1} - \frac{a^2}{4} + j_1\frac{\psi_0-\psi_c}{j_1+1}\ln\frac{4(\psi_0-\psi_c)}{a(j_1+1)}, & a \in [a_c, 1] \\ -\ (j_1+1)\frac{a^2}{4} + \psi_0, & a \in [0, a_c], \end{cases} \tag{5}$$

where a_c – the value of the argument, which corresponds to ψ: $\psi(a_c) = \psi_c$. The value of $\psi_0 = \psi(0) > \psi_c$ – the value of ψ in the center – is defined by the equation

$$\psi_c + \frac{\psi_0 - \psi_c}{j_1 + 1} - \frac{1}{4} + j_1\frac{\psi_0 - \psi_c}{j_1 + 1}\ln\frac{4(\psi_0 - \psi_c)}{(j_1 + 1)} = 0. \tag{6}$$

The number of solutions to the problem (4) is determined by the number of roots ψ_0 to the algebraic equation (6), and the existence of the parabolic solution (note that the parabolic solution disappears if $\psi_c \leq 1/4$). For any non-negative j_1, the equation (6) at $\psi_0 > \psi_c$ gives an implicit determination of the curve $\psi_0(\psi_c)$, which transforms into the straight line $\psi_0 = 1/4$ at $\psi_0 \leq \psi_c$. This curve is depicted in Fig. 2 for $j_1 = 1.5$. In the hatched area there are three roots ψ_0 for every value of ψ_c, outside this area - only one, on the boundary - two.

Fig.3 represents the multiple solutions at the following values of parameters: $\psi_c = 0.28$, $j_1 = 1$.

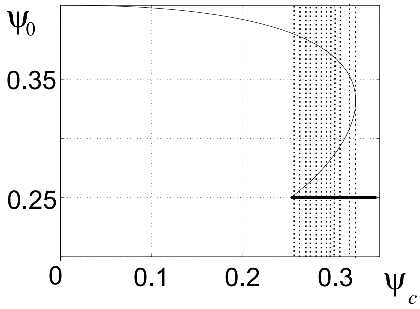

Fig. 2. Dependence (6) $\psi_0 = \psi_0(\psi_c)$, $j_1 = 1.5$. The area of multiple solutions is ticked off by points.

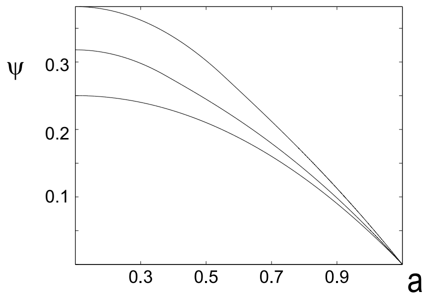

Fig. 3. The solutions of boundary problem (4) calculated for the following parameters values: $\psi_c = 0.28$, $j_1 = 1$.

The multiple solutions to the equation of equilibrium found in cylindrical approximation are obvious to have their analogues in toroidal case. It can be easily proved by use of moment representation for the magnetic surfaces [7]. For circlular magnetic surfaces with large but finite aspect ratio, $\lambda = R/a_b \gg 1$, the main toroidal effect reveals itself by an appearance of the Shafranov shift, $\Delta(a)$. It's essential that the multiple solution found above do not violate the conventional ordering, $\Delta/a_b \sim 1/\lambda \ll 1$, and conserve the nesting of the magnetic surfaces. Hence, our multiple solutions correspond to different equilibrium states in toroidal geometry as well. We present an example of calculation of $\Delta(a)$ for $p(\psi)$ chosen to be linear function with $p'(\psi) = 0.1$ (the pressure is made dimensionless by dividing by $j_0^2 R^2$). Taking $j_1 = 1.5$, $\psi_c = 0.255$, we found that the boundary problem (4) has three solutions. Two of them are shown in Fig.4 as a set of poloidal cross-sections $\psi^{(1)} = \text{const}$ and $\psi^{(3)} = \text{const}$ by the plane (r, z). Solid and dotted contours correspond to the magnetic surfaces of different solutions ($\psi^{(1)}(r, z)$, and $\psi^{(3)}(r, z)$ correspondingly), which have the same value of the flux ψ ($R = 1$, $\lambda = 3$).

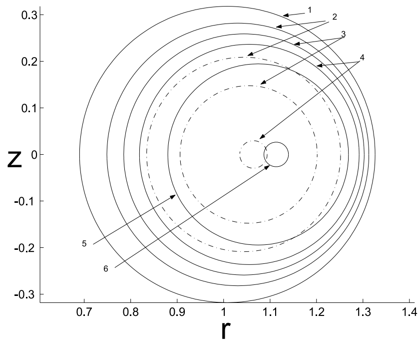

Fig. 4. Sections of surfaces $\psi^{(1,3)}(r,z)$ by values:1. $\psi = 0$; 2. $\psi = 0.15$; 3. $\psi = 0.2$; 4. $\psi = 0.248$; 5. $\psi = 0.35$; 6. $\psi = 0.563$. Dotted line corresponds to $\psi^{(1)}(r,z)$, solid line – to $\psi^{(3)}(r,z)$.

We have shown that Grad-Shafranov equation *in its classical treatment*, i.e., for given dependencies $p(\psi)$, $F(\psi)$, can have several different solutions, satisfying *the same* boundary conditions. This result was obtained analytically in cylindrical approximation and was confirmed numerically for the toroidal geometry. The number of solutions undergoes spontaneous changes while the parameters in the equation change continuously, therefore, it may be called "a bifurcation".

It should be noted that we have demonstrated the bifurcational behavior of the equilibria for a very simple (but nonlinear) current density profile. Hence, it's probable to expect that multiple solutions are more typical for plasma equilibrium than the unique solution, which takes place, e.g., for a degenerated situation with linear ψ – function in the right-hand-side of Eq.(1).

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