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MHD Stability of Toroidal Configurations in the Limit of Vanishing Rotational Transform

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For flute oscillations to be stable in toroidal configuration with closed magnetic field lines and with a zero rotational transform, μ, it is necessary and sufficient to satisfy Bernstein-Kadomtsev (BK) condition [1,2]

$$-\left(\oint Bdl\right)(\nabla U)^2 U^{-2} < \nabla p \cdot \nabla U < \gamma_0 p (\nabla U)^2 |U|^{-1} . \quad (1)$$

Here, p is the plasma pressure, γ_0 is the adiabatic exponent, $U = -\oint dl/B$ is the label of the equilibrium magnetic surface. For configurations with a magnetic well and decreasing plasma pressure, $\nabla p \cdot \nabla U < 0$, the right-hand inequality in (1) holds, so that the problem is to satisfy the left-hand inequality. Otherwise, for systems with a magnetic hill, $\nabla p \cdot \nabla U > 0$, the left-hand inequality in (1) holds, and the problem is to satisfy the right-hand inequality. There exists a marginal stable profile, $p \sim |U|^{-\gamma_0}$, which reduces the right-hand inequality in (1) to an identity. This possibility of MHD stabilization is the main subject of our study.

The assumption that the magnetic field lines are closed over the entire volume of a confinement system is a theoretical idealization. In a real situation, the field lines are unclosed because of the possible magnetic field distortions. In analyzing the MHD stability of systems with closed magnetic field lines, it is logical to consider how it is affected by a small distortion that makes the lines slightly unclosed.

For toroidal systems with a nonzero rotational transform, the Mercier criterion is known. The Mercier criterion predicts that a system with a small shear and without a magnetic well is unstable. Mikhailovskii and Skovoroda [3] showed that taking into account plasma compressibility can substantially change the growth rate of MHD instability that develops when the Mercier criterion is violated. Under an inequality similar to the right-hand inequality in (1), the instability grows not at the rate of development of Alfven modes, $\gamma \sim c_A / L_s$, but at the rate of development of acoustic modes, $\gamma \sim c_s / L_s$, which is substantially slower at low β. Here L_s is the shear length. But the question of how the instability growth rate behaves when the shear approaches zero remained open.

We shall show that in ideal MHD approximation the increment remains nonzero in the limit of vanishing rotational transform. The continuos proceeding to the limit $\mu = 0$ will be obtained taking into account finite Larmor radius effect.

1. *Cylinder geometry*

The familiar exact equation for ideal small radial displacements, ξ, in cylinder geometry can be represented in the form

$$\frac{C_{12}}{r}\frac{d}{dr}\left(\frac{r^3\rho\left(\gamma^2+k_\parallel^2c_A^2\right)}{C_{12}}\frac{d\xi}{dr}\right)-W\xi=0, \tag{2}$$

$$W=\rho\left(\gamma^2+k_\parallel^2c_A^2\right)\left(m^2-1+k_z^2r^2-\chi r^2\right)+2rk_z^2\frac{dp}{dr}+\frac{4rk_z^2\rho c_s^2}{R}\frac{\gamma^2}{\gamma^2(1+\beta)+k_\parallel^2c_s^2}-\gamma^2r\frac{d\rho}{dr}-$$

$$-2B_\varphi\frac{d(rB_\varphi\chi)}{dr}+\frac{r}{C_{12}}\frac{dC_{12}}{dr}\left(\rho(\gamma^2+k_\parallel^2c_A^2)-C_{11}\right),\quad C_{12}=-m^2-k_z^2r^2+r^2\chi,$$

$$C_{11}=2\left(\frac{mB_\varphi}{r}k_\parallel B-B_\varphi^2\chi\right),\ \chi=-\frac{\gamma^4}{c_A^2\left(\gamma^2(1+\beta)+k_\parallel^2c_s^2\right)},\ \beta=\frac{c_s^2}{c_A^2},\ \frac{1}{R}=\frac{B_\varphi^2}{rB^2}.$$

Here R^{-1} is the magnetic field line curvature, k_z is the wave number along the cylinder axis, m is the azimuthal mode number, and $B_{z,\varphi}$ are the magnetic field components. The plasma compressibility is explicitly accounted for by the third term in the expression for W.

As a model of a confinement system with poloidally closed magnetic field lines, we use a straight cylinder in which the magnetic field lines have the only nonzero component B_φ. In this case (other cases see in [4]), the effect of distortions that make the field lines unclosed is modeled by a weak uniform magnetic field $B_z << B_\varphi$. We consider $\beta << 1$ plasma, and mode $m = 0$ (at $B_z = 0$ longitudinal wave number $k_\parallel = 0$ only for $m = 0$). Since BK condition (1) is fulfilled, $p' + 2\frac{\gamma_0 p}{r} > 0$, the growth rate is estimated to be $\gamma^2 \sim k_\parallel^2 c_s^2 << k_\parallel^2 c_A^2$, and we can assume that $k_z^2 >> \chi$ to obtain $C_{12} \sim -k_z^2 r^2$, $C_{11} \sim 0$. Since $k_\parallel^2 = k_z^2 B_z^2 / B^2$ and $k_z \neq 0$, we divide (2) by k_z^2 and arrive at the equation

$$B_z^2 r\left(r\xi'\right)' - \xi\left(B_z^2(1+k_z^2r^2)+2rp'+\frac{4\gamma_0 p\gamma^2}{\gamma^2+k_\parallel^2c_s^2}\right)=0. \tag{3}$$

From (3) we obtain the following estimate for the upper limit of the increment, γ, at a fixed wave number k_z [4]:

$$\gamma^2 \sim -k_z^2 B_z^2 \int_0^a \xi^2 \frac{c_s^2}{B^2} p' dr \Bigg/ \int_0^a \xi^2 \left(p' + 2\frac{\gamma_0 p}{r} \right) dr . \tag{4}$$

We can see that the increment approaches zero as B_z approaches zero, and it increases with k_z. However, the product $k_z B_z$ is restricted by the stabilization condition $k_z^2 B_z^2 \sim -p'/r$, which leads to the maximum possible growth rate $\sim \sqrt{\beta}\gamma_A$ (γ_A is the usual flute mode increment). In this approximation, for any small B_z we can find so large k_z to obtain the finite increment of instability.

2. *General geometry*

We use the familiar ideal ballooning equations for small-scale modes

$$\begin{aligned} &\left[\mathbf{B}\cdot\nabla\left(k_\perp^2 k_b^{-2}\mathbf{B}\cdot\nabla\right) + 2p'R^{-1} + \rho\omega^2 k_\perp^2 k_b^{-2}\right]\xi = -2\gamma_0 pXR^{-1}, \\ &\left[\mathbf{B}\cdot\nabla\left(c_s^2\omega^{-2}B^{-2}\mathbf{B}\cdot\nabla\right) + \left(\gamma_0 p + B^2\right)B^{-2}\right]X = -2\xi R^{-1} \end{aligned} . \tag{5}$$

Here ξ and $X = \nabla\cdot\boldsymbol{\xi}$ are the normal component and divergence of displacement vector $\boldsymbol{\xi}$, $\mathbf{k}_\perp$ is the wave vector, k_b is the azimuthal component of $\mathbf{k}_\perp$, $R^{-1} = [\mathbf{B}\times\boldsymbol{\kappa}]\cdot\mathbf{k}_\perp/Bk_b$, $\boldsymbol{\kappa}$ is the vector of field line curvature, prime denotes the derivative with respect to the label of the magnetic surface, ρ is the mass density.

We shall use the magnetic coordinates a, β, ζ for magnetic field representation

$$2\pi\mathbf{B} = \Phi'[\nabla a \times \nabla\beta], \quad 2\pi\mathbf{B} = F\nabla\zeta - \nu\nabla a . \tag{6}$$

Here $\beta = \theta - \mu\zeta$, $\mu = -\Psi'/\Phi'$ is rotational transform, θ and ζ are the angle Boozer coordinates. The $\mu \to 0$ limit can be investigated at $\mu' = 0$ approximation (very small shear does not change the final conclusion). Using the Fourier transform of ξ (and X)

$$\xi = \sum_{n,m=-\infty}^{\infty} \xi_{n,m} e^{i(n\zeta - m\theta)} = \sum_{n,m=-\infty}^{\infty} \xi_{n,m} e^{i(n-\mu m)\zeta} e^{-im\beta} = \sum_{m=-\infty}^{\infty} \overline{\xi}_m(\zeta) e^{-im\beta} , \tag{7}$$

we obtain for almost-periodic function $\overline{\xi}_m(\zeta)$ the discrete spectrum $k_{zn} = n - \mu m$. In [4] it was shown that "anti-Mercier", $m >> 1$, perturbations are the worst case for stability at $\mu \to 0$. For quasi-flute perturbations the system (5) gives the dispersion relation

$$U_0 + U_1\left(1 + k_z^2/k_{zs}^2\right)^{-1} + k_{zA}^2 + k_z^2 = 0 . \tag{8}$$

Here U_0 and U_1 are the familiar values of magnetic well/hill and compression [3,4], $k_{zs} \sim \gamma L/c_s$, $k_{zA} \sim \gamma L/c_A$. When BK condition is fulfilled, $U_0 + U_1 \ge 0$, ($U_0 < 0$) we obtain from (8) the condition of stability

$$0 < k_z^2 < -U_0 \qquad \text{or} \qquad 0 < (n - \mu m)^2 < -U_0 . \tag{9}$$

At $\mu = 0$ the flute perturbations, $n = 0$, are stable. The ballooning mode, $n = 1$, is unstable at β, determined by the condition $-U_0 > 1$. At any $\mu << 1$ we can always find such $m >> 1$ that $\mu \sim n/m << 1$, and we obtain the instability with maximal increment, determined by relation $k_{zs\,\max}^2 = \sqrt{U_1} - \sqrt{U_0 + U_1}$.

3. *Account of finite Larmor radius*

The finite Larmor radius effect leads to modification of equations (6):

$$\begin{aligned} &\left[\mathbf{B}\cdot\nabla\left(k_\perp^2 k_b^{-2}\mathbf{B}\cdot\nabla\right) + 2p'R^{-1} + \rho\omega(\omega - \omega^* + 1.5\omega_R)k_\perp^2 k_b^{-2}\right]\xi = -2\gamma_0 pXR^{-1}, \\ &\left[\mathbf{B}\cdot\nabla\left(c_s^2\omega^{-1}(\omega - \omega_M)^{-1}B^{-2}\mathbf{B}\cdot\nabla\right) + (\gamma\, p + B^2)B^{-2}\right]X = -2\xi R^{-1}(\omega - \omega^*)\omega^{-1} \end{aligned} \tag{10}$$

Here $\omega^* = k_b p'/\omega_{ci}\rho$ is the diamagnetic drift frequency, $\omega_M = \omega_B + \omega_R$ is the magnetic drift frequency, $\omega_R = pk_b/R\omega_{ci}\rho$, $\omega_B = pk_b/R_B\omega_{ci}\rho$, $R_B^{-1} = \mathbf{k}_\perp \cdot [\mathbf{B}\times\nabla B]/k_b B^2$. As in Section 2 we can obtain the dispersion relation for quasi-flute perturbations. The analysis shows the existence on instability only at the inequality

$$(n - \mu m)^2 > U_1 \langle\omega^*\rangle^2 / 4(U_0 + U_1)\langle\omega_s^2\rangle , \tag{11}$$

$\langle\ \rangle$ denotes the average values of drift and sound frequencies. At $n = 0$ the stability is realized up to rotational transform (this is the lower estimation for $n \neq 0$ also)

$$\mu_l \sim \sqrt{U_1 \langle\omega^*\rangle^2 / 4(U_0 + U_1)\langle\omega_s^2\rangle m^2} \approx \rho_i R_0 / a^2 . \tag{12}$$

Here a and R_0 are the minor and major radii of a torus, ρ_i is the average Larmor radius.

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