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Constructive and rigorous approach to the nonlinear tearing mode

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Abstract. Within the frame of standard 2D reduced MHD, the time evolution equation for a nonlinear tearing island width w is computed rigorously for a general current gradient at orders $w \ln(1/w)$ and w. Several technical improvements with respect to previous works enable this to be done with reasonable algebra and without any Ansatz.

Introduction. A fundamental problem in magnetohydrodynamics (MHD) is the determination of the nonlinear evolution and of the saturation amplitude of tearing modes. Rutherford [2] found that the linear stage, where the island growth is exponential, is followed by a nonlinear one where the island width w grows linearly with time on the resistive timescale. Later on Biskamp made rigorous this calculation [3]. The saturation of the island width was described in the same limit by White *et al.* [4] who gave a criterion for saturation predicting w to be proportional to Δ'. For non-zero current gradients, a calculation of Thyagaraja [5] showed that $\Delta' \sim w \ln(1/w)$ and computed the corresponding coefficient. For zero current gradients, Refs. [6] and [7] showed independently that $\Delta' \sim w$ and computed the corresponding coefficient. This was done rigorously in Ref [7] which further gave the time evolution equation for w.

This work yields the time evolution equation for w for a general current gradient, by using the standard 2D reduced MHD equations for a sheet pinch. Several technical improvements with respect to previous works enable the rigorous calculations of orders $w \ln(1/w)$ and w with reasonable algebra: (i) using flux-angle coordinates enables to stick to the geometrical structure of the magnetic island; this brings a strong simplification of the algebra, in particular by a simultaneous treatment of the trapped and untrapped domains, (ii) no Ansatz is made on the solution, which simplifies the notations; the non vanishing orders $w^m[\ln(1/w)]^n$ show up successively. The evolution equation for w, as a rule, can be computed to arbitrary order in the $w^m[\ln(1/w)]^n$ expansion. The first two orders $w \ln(1/w)$ and w are explicitly computed, since they provide contributions with the same order of magnitude for practical purposes. The calculation combines the Rutherford and saturation regimes, and shows that the current profile has the same algebraic structure in both regimes.

Model. As is usual when dealing with the nonlinear tearing mode [2, 3], inertia and viscosity are neglected in the 2D reduced MHD equations, which leads to the basic equations

$$[\psi, J] = 0, \tag{1}$$
$$\partial_t\psi + [\varphi, \psi] = 1 - \eta(x)J, \tag{2}$$
$$\nabla^2\psi = -J, \tag{3}$$

where ψ is the magnetic flux function, φ is the electric potential which plays the role of the stream function, and $\eta(x) = 1 + \sum_{i=1}^{\infty} a_i x^i$ is the resistivity profile. Moreover, for any two fields A and B, $[A, B] = \partial_x A \partial_y B - \partial_y A \partial_x B$. We consider a perturbation to a static equilibrium defined by $\varphi = 0$ and $J = J_0(x)$ such that $\eta(x)J_0(x) = 1$ and $\psi_0'(0) = 0$. x and y may be viewed as the radial and poloidal coordinates in a tokamak (where $x = 0$ corresponds to a given finite radius). Accordingly we assume periodicity in y. Lengths and times are normalized, as usual, to a suitable macroscopic scalelength a, and to the corresponding Alfven time $\tau_A = a/V_A$, where V_A is the Alfven speed.

We consider a tearing magnetic island corresponding to the $x = 0$ magnetic flux surface, with a wavenumber k in the y direction, and a normalized width w scaling like a small parameter δ. In the weakly nonlinear regime of the tearing mode, we expect a boundary layer centered on $x = 0$ to scale similarly. The rest of the system is called the outer domain. In the following instead of x and y we use the stretched 'radius' $\xi = x/\delta$ and $\chi = ky$, and we define a rescaled magnetic flux $\zeta = -\psi/\delta^2$. Equations (1), (2), and (3) respectively become

$$[\zeta, J]_{(\xi,\chi)} = 0, \tag{4}$$
$$\delta^2\partial_t\zeta + k\delta\,[\zeta, \phi]_{(\xi,\chi)} = 1 - \left(1 + \delta a_1\xi + \delta^2 a_2\xi^2 + O(\delta^3)\right) J, \tag{5}$$
$$\partial_\xi^2\zeta + \delta^2 k^2 \partial_\chi^2\zeta = J, \tag{6}$$

where $[A, B]_{(\xi,\chi)} \equiv \partial_\xi A\, \partial_\chi B - \partial_\chi A\, \partial_\xi B$.

Method of solution.

As usual for the nonlinear tearing mode, the outer system is described by linearizing (1). This provides an expansion for ξ small of the order 0 and 1 Fourier harmonics of ζ_{out} in χ

$$\zeta_{out} \sim \frac{\xi^2}{2} - \cos\chi - \delta\ln\delta b_1\xi\cos\chi + \delta\left(\frac{b_1}{6}\xi^3 - \left[\frac{1}{2}\Sigma'\xi + \frac{1}{2}\Delta'|\xi| + b_1\xi\ln|\xi|\right]\cos\chi + \mu^{\pm}\right) \tag{7}$$

where Σ' and Δ' are the sum and difference of the logarithmic derivatives of ζ_{out} at $\xi = \pm 0$; Δ' is the usual tearing mode stability parameter. The constant $\mu^{\pm}$ is introduced because the inner solution (that inside the boundary layer) turns out to bring in a jump of magnetic flux between the left and right outer domains.

Since Eq. (4) implies that $J(\xi,\chi) = j\,(\zeta(\xi,\chi))$, it is natural to perform the change of variables $(\xi,\chi) \to (\zeta,\chi)$. As may be anticipated from the linear regime, this change of variables is not one to one, and the following analysis shows that a given value of ζ corresponds to two different values of ξ defined by $\xi = X^{\pm}(\zeta,\chi)$ where $X^{\pm}$ is computed below. The $X^{\pm}$'s define in the plane (ξ,χ) two regions $\mathcal{D}^-$ and $\mathcal{D}^+$ separated by a curve close to $\xi = 0$. Accordingly we define $\Phi^{\pm}(\zeta,\chi) = \phi[X^{\pm}(\zeta,\chi),\chi]$. Then in the stationary case Eq. (5) becomes

$$k\delta\,\partial_\chi \Phi^{\pm}(\zeta,\chi) = \partial_\zeta X^{\pm}\left\{1 - \left(1 + a_1\delta X^{\pm} + a_2\delta^2 (X^{\pm})^2 + O(\delta^3)\right) j^{\pm}(\zeta)\right\}, \qquad (8)$$

where $j^{\pm}(\zeta)$ is the value of $J(\zeta)$ in the $\mathcal{D}^{\pm}$ domains. For δ small our analysis shows that the nonlinear tearing mode corresponds to a magnetic island topologically equivalent to that of the linear regime. Therefore the curves $\zeta =$ constant correspond to curves $\mathcal{C}_\zeta$ with two possible topologies corresponding to trapping and untrapping in the magnetic island. Integrating (8) on $\mathcal{C}_\zeta$ and using the fact that $\Phi^{\pm}$ is 2π-periodic in χ and single-valued on the curve limiting $\mathcal{D}^-$ and $\mathcal{D}^+$, we obtain

$$j^{\pm}(\zeta) = F_0^{\pm}(\zeta) \times \left(F_0^{\pm}(\zeta) + a_1\delta F_1^{\pm}(\zeta) + a_2\delta^2 F_2^{\pm}(\zeta) + O(\delta^3)\right)^{-1}, \qquad (9)$$

where

$$F_n^{\pm}(\zeta) = \int_{\mathcal{C}_\zeta} (X^{\pm})^n \partial_\zeta X^{\pm} d\chi. \qquad (10)$$

Equation (9) was already introduced in [5]. Its structure suggests to solve Eqs. (6) and (9) for ζ by perturbation expansion in δ. This is done iteratively in a series of steps which are now illustrated for the lowest order calculation. At this order Eq. (9) yields $j_0^{\pm} = 1$. Then Eq (6) provides $\zeta_0(\xi,\chi) = \frac{1}{2}\xi^2 + A(\chi)\xi + B(\chi)$ where $A(\chi)$ and $B(\chi)$ are two unknown functions which are now determined by matching with ζ_{out}. Since the next order in the expansion of ζ provided by Eqs. (6) and (9) is δ, the matching brings to $A(\chi)$ and $B(\chi)$ terms of order 1 and $\delta \ln \delta$. As a result $\zeta_0 = \zeta_{00} + \delta \ln \delta \zeta_{11}$ where $\zeta_{00}(\xi,\chi) = \frac{1}{2}\xi^2 - \cos\chi$ and $\zeta_{11}(\xi,\chi) = a_1\xi\cos\chi$. Then the leading order $X_0^{\pm}$ of $X^{\pm}$ can be computed by solving $\zeta = \zeta_0(X_0^{\pm}(\zeta,\chi),\chi) + \delta \ln \delta \zeta_{11}(X_0^{\pm}(\zeta,\chi),\chi)$ at orders 1 and

$\delta \ln \delta$ for $X_0^\pm$. This yields $X_0^\pm(\zeta,\chi) = X_{00}^\pm(\zeta,\chi) + \delta \ln \delta X_{11}(\zeta,\chi)$ where $X_{00}^\pm = \pm X_*(\zeta,\chi)$ with $X_*(\zeta,\chi) = [2(\zeta + \cos\chi)]^{1/2}$, and $X_{11} = -a_1 \cos\chi$. This ends the first iteration of the perturbation calculation. We notice that indeed two values of ξ are related to one value of ζ. The next iteration starts by setting $X_0^\pm$ in Eq. (9), which brings orders $\delta j_{10}^\pm(\zeta) + \delta^2 \ln \delta j_{21}^\pm(\zeta)$ to $j^\pm$. These orders are brought in Eq. (6), which brings contributions $\delta\zeta_{10} + \delta^2 \ln \delta \zeta_{21}$ to ζ, and so on.

The matching of the inner and outer ζ's shows that Δ' is at most of order $\delta \ln(1/\delta)$. Indeed ζ_{21} brings to Δ' a contribution $\delta \ln(1/\delta)\Delta'_{11}$ which was already computed in [5]. The next contribution in the expansion of Δ' is brought by a term $\delta^2\zeta_{20}$ in the expansion of ζ and is of order δ. It contains a term proportional to a_2 which was computed in Refs. [6, 7]. This analysis can be simply generalized to the time varying case and yields for the island width $w = 4\delta$ an equation of the type

$$\frac{\mathrm{d}w}{\mathrm{d}t} = 1.22\eta[\Delta' - 0.41\, a_1^2 w \ln(1/w) - w(0.41\, a_2 + c_1 a_1 \Sigma' + c_2 a_1^2)], \tag{11}$$

where η is the scaled resistivity at $x = 0$ and the c_i's are constants given by definite integrals. A similar result holds in the case of a constant resistivity and a non constant $J_0(x)$.

References

[1] H.P. Furth, J. Killeen, M.N. Rosenbluth, Phys. Fluids 6 (1963) 459.

[2] P. Rutherford, Phys. Fluids 16 (1973) 1903.

[3] D. Biskamp, *Nonlinear Magnetohydrodynamics* (Cambridge University Press,Cambridge UK,1993)

[4] R.B. White, D.A. Monticello, M.N. Rosenbluth, B.V. Waddell, Phys. Fluids 20 (1977) 800.

[5] A. Thyagaraja, Phys. Fluids 24 (1981) 1716.

[6] F. Militello, F. Porcelli, Phys. Plasmas 11 (2004) L13.

[7] D.F. Escande, M. Ottaviani, Phys. Lett. A 323 (2004) 278.