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Resistive Wall Modes and Error Field Amplification

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The theory of Maxwell's and Laplace's equations allows a relatively simple analysis of wall modes and error field amplification in terms of the properties of plasma inductance matrices. Both phenomena can be characterized by the effect of the perturbed plasma current on an externally imposed magnetic perturbation. Here, the linear theory of these phenomena is developed, and the calculation of the inductance matrices are discussed for a plasma that obeys the constraints of ideal magnetohydrodynamics.

I. INTRODUCTION

Resistive wall modes [1–5] are kink instabilities that are stabilized by the conducting structures that surround a plasma. Resistive wall modes grow on the resistive time scale of the structures, which is sufficiently slow that these modes can be feedback stabilized. A magnetic field error that resonates with a natural mode of the plasma is increasingly amplified by the perturbed plasma currents as the mode approaches marginal stability [6, 7]. In a non-rotating plasma, the error field amplification goes to infinity at marginal stability. Both phenomena can provide important limitations on the performance of tokamaks and can be characterized by the effect of the perturbed plasma current on an externally imposed magnetic perturbation. Here we discuss the theory and the calculation of the inductance matrices that characterize the effects of the perturbed plasma current on the external magnetic field.

The effect of a plasma on an externally driven magnetic perturbation is defined by the relation, at the location of the perturbed plasma surface, between the perturbed normal magnetic field $\hat{n}\cdot\delta\vec{B}$ and the scalar potential ϕ of the perturbed tangential magnetic field. If $\vec{x}_s(\theta,\varphi)$ is the equation for the unperturbed plasma surface, ϕ is related to the tangential components of the magnetic field by $\partial\phi/\partial\theta = \vec{B}\cdot\partial\vec{x}_s/\partial\theta$ and $\partial\phi/\partial\varphi = \vec{B}\cdot\partial\vec{x}_s/\partial\varphi$. If the normal magnetic field $\hat{n}\cdot\delta\vec{B}$ and the magnetic potential ϕ are expanded in othonormal functions on the plasma surface, $f_i(\theta,\varphi)$, then inductance matrices relate the two sets of expansion coefficients to each other.

To completely characterize perturbed magnetic fields outside of the plasma, the only properties of the plasma that are required are [5] the shape of the unperturbed plasma surface, $\vec{x}_s(\theta,\varphi)$; the inductance matrix of that surface in the presence of the plasma, $\overleftrightarrow{\Lambda}$; and a purely geometric quantity, the inductance matrix of the plasma surface in the absence of a plasma $\overleftrightarrow{L}$. Once these properties are defined, the VALEN code [1] provides a complete multimode model of the rotating plasma response to resistive wall modes, feedback systems, and error fields.

II. FLUXES AND CURRENTS

The spatial distribution of the normal magnetic field on a toroidal surface, such as the plasma surface, can be specified in terms of a set of magnetic fluxes Φ_i, and the distribution of current flowing on the surface can be specified in terms of a set of currents J_j. This section will explain how this is done. The inductance matrix of the surface is the linear relation between the fluxes and the currents, $\Phi_i = \sum_j L_{ij}J_j$, which can also be written as $\vec{\Phi} = \overleftrightarrow{L}\cdot\vec{J}$.

A toroidal plasma surface can be written as $\vec{x}_s(\theta,\varphi)$; the unit normal to the surface is

$$\hat{n} = \frac{\frac{\partial\vec{x}_s}{\partial\theta}\times\frac{\partial\vec{x}_s}{\partial\varphi}}{\left|\frac{\partial\vec{x}_s}{\partial\theta}\times\frac{\partial\vec{x}_s}{\partial\varphi}\right|}. \quad (1)$$

Coordinates (r,θ,φ) defined by $\vec{x}(r,\theta,\varphi) = \vec{x}_s(\theta,\varphi) + (r-a)\hat{n}$ have the Jacobian $\mathcal{J} = \left|\frac{\partial\vec{x}_s}{\partial\theta}\times\frac{\partial\vec{x}_s}{\partial\varphi}\right|$. A dual relation of the theory of general coordinates says that $\vec{\nabla}r = \left(\frac{\partial\vec{x}_s}{\partial\theta}\times\frac{\partial\vec{x}_s}{\partial\varphi}\right)/\mathcal{J}$, which implies $\vec{\nabla}r = \hat{n}$.

Given a toroidal surface $\vec{x}_s(\theta,\varphi)$ and a set of othonormal functions on that surface, $\oint f_i(\theta,\varphi)f_j^* da = A\delta_{ij}$, with $A \equiv \oint da$, the normal magnetic field through the surface of the plasma and the current on that surface can be expanded in terms of the $f_i(\theta,\varphi)$. The normal magnetic field through the plasma surface is

$$\vec{B}\cdot\hat{n} = \frac{1}{A}\sum_i \Phi_i^* f_i(\theta,\varphi). \quad (2)$$

The expansion coefficients $\Phi_i = \oint f_i\delta\vec{B}\cdot d\vec{a}$ have units of magnetic flux.

The most general expression for a divergence-free vector in a toroidal surface, such as the current flowing on the plasma surface at $r = a$, is

$\vec{j} = \delta(r-a)\vec{\nabla}\kappa \times \vec{\nabla}r$, which can be written using the theory of general coordinates as

$$\vec{j} = \left(\frac{\partial \kappa}{\partial \varphi} \frac{\partial \vec{x}_s}{\partial \theta} - \frac{\partial \kappa}{\partial \theta} \frac{\partial \vec{x}_s}{\partial \varphi} \right) \frac{\delta(r-a)}{\mathcal{J}}. \quad (3)$$

The current potential $\kappa(\theta, \varphi)$ has units of Amperes and can be expanded in terms of current J_j as

$$\kappa = \sum_j J_j f_j^*(\theta, \varphi). \quad (4)$$

The effect of external currents on the plasma can be represented by currents flowing on a control surface located an infinitesimal distance outside of the plasma. The magnetic field between the plasma, conducting walls, and the coils is both curl and divergence free, so that $\vec{B} = \vec{\nabla}\phi$ with $\nabla^2\phi = 0$ in that region. In the presence of the control surface, the magnetic field outside of the plasma obeys the equation $\vec{\nabla} \times \vec{B} = \mu_0 \vec{\nabla} \times (\kappa\delta(r-a)\vec{\nabla}r)$ with κ the current potential on the control surface. This equation implies $\vec{B} = \vec{\nabla}\phi + \mu_0\kappa\delta(r-a)\vec{\nabla}r$, so the jump in the magnetic potential across the control surface is $[\phi] = -\mu_0\kappa$. The equation $\vec{\nabla} \cdot \vec{B} = 0$ implies $[\hat{n} \cdot \vec{\nabla}\phi] = 0$. The normal magnetic field on the control surface determines the magnetic potential ϕ in the current-free region external to the control surface. Therefore, specifying $\hat{n} \cdot \vec{B}$ and ϕ on the plasma surface is equivalent to specifying $\hat{n} \cdot \vec{B}$ and κ on the control surface.

Any code that can study perturbed plasma states, such as the ideal magnetohydrodynamics (MHD) code DCON [8], provides the relation between $\hat{n} \cdot \vec{B}$ and ϕ on the plasma surface and can be used to determine the inductance matrices $\overleftrightarrow{L}$ and $\overleftrightarrow{\Lambda}$ of the plasma surface without and with a plasma present.

III. PLASMA INTERACTION WITH EXTERNAL CURRENTS

The perturbed magnetic fields outside of the plasma depend on the plasma only through the shape of the unperturbed plasma surface, $\vec{x}_s(\theta, \varphi)$; the inductance matrix of that surface in the presence of the plasma, $\overleftrightarrow{\Lambda}$; and a purely geometric quantity, the inductance matrix of the plasma surface in the absence of a plasma $\overleftrightarrow{L}$. The perturbed magnetic field is determined by currents in external circuits, which can be represented by a matrix vector $\vec{I}_w$, and by the perturbed current density $\delta\vec{j}_p$ in the plasma. The effect of $\delta\vec{j}_p$ on the external fields is to modify the magnetic scalar potential ϕ that is associated with a given $\delta\vec{B} \cdot \hat{n}$ on the plasma surface. Outside of the plasma surface, this effect can be represented by a surface current with coefficients $\vec{I}_p$ on the plasma surface.

The effect of the perturbed plasma current on the magnetic field outside of the plasma can be represented by a vector of current coefficients $\vec{I}_p$ on the plasma surface. The effective perturbed plasma current is

$$\vec{I}_p = \overleftrightarrow{L}^{-1} \cdot (\overleftrightarrow{\Lambda} - \overleftrightarrow{L}) \cdot \overleftrightarrow{L}^{-1} \cdot \vec{\Phi}_x, \quad (5)$$

with $\vec{\Phi}_x$ the vector of matrix coefficients of the normal magnetic field on the plasma surface due to external currents. This equation is easily proven by noting that the coefficients of the total normal field on the plasma surface is $\vec{\Phi} = \vec{\Phi}_x + \vec{\Phi}_p$, where the normal field produced by external currents is described by $\vec{\Phi}_x = \overleftrightarrow{L} \cdot \vec{J}$ and the normal field produced by plasma currents is described by $\vec{\Phi}_p = \overleftrightarrow{L} \cdot \vec{I}_p$. The total normal field is related to $\vec{J}$ by $\vec{\Phi} = \overleftrightarrow{\Lambda} \cdot \vec{J}$. Trivial algebra gives the relation between $\vec{\Phi}_x$ and $\vec{I}_p$.

Once the effect of the perturbed plasma currents, $\delta\vec{j}_p$, is known, the forces $\vec{F}$ between the plasma and the external structures, which carry a current $\delta\vec{j}_w$, are known since $\vec{F} = \int_w \delta\vec{j}_w \times \delta\vec{B}_p d^3x$. The power required to drive each external current is also known since $-\int_w \delta\vec{j}_w \cdot \vec{E} d^3x = \int_w \delta\vec{B}_w \cdot \partial\delta\vec{B}/\partial t d^3x$ with the use of the Faraday's law. The perturbed magnetic field is $\delta\vec{B} = \delta\vec{B}_w + \delta\vec{B}_p$, where $\delta\vec{B}_w$ is the perturbed field due to the currents in external structures, $\delta\vec{j}_w$, and $\delta\vec{B}_p$ is the magnetic field due to the perturbed plasma current, $\delta\vec{j}_p$, or, equivalently, the field due to a current on the plasma surface with expansion coefficients $\vec{I}_p$.

Currents outside of the plasma can be represented as a set of discrete circuits, each with a flux $(\Phi_w)_i$ and a current $(I_w)_i$. For explicit coils, the definition of $(\Phi_w)_i$ and $(I_w)_i$ is obvious. Thin, continuous structures, such as walls, are represented in a manner analogous to the surface of the plasma. That is, in terms of expansion coefficients for the normal magnetic field through the surface and for the current potential that represents the current flowing in the structure. The magnetic flux through circuits external to the plasma has the form $\vec{\Phi}_w = \overleftrightarrow{L}_w \cdot \vec{I}_w + \overleftrightarrow{M}_{wp} \cdot \vec{I}_p$. Both, the inductance $\overleftrightarrow{L}_w$, and the mutual inductance matrix between the wall and the plasma $\overleftrightarrow{M}_{wp}$, are calculated by routines in the VALEN computer code [1]. The effective perturbed plasma current $\vec{I}_p$ can be calculated using Equation (5) by writing $\vec{\Phi}_x = \overleftrightarrow{M}_{pw} \cdot \vec{I}_w$. One can write $\vec{\Phi}_w = \overleftrightarrow{\mathcal{L}}_w \cdot \vec{I}_w$, where the effective inductance of the wall is given by

$$\overleftrightarrow{\mathcal{L}}_w \equiv \overleftrightarrow{L}_w + \overleftrightarrow{M}_{wp} \cdot \overleftrightarrow{L}^{-1} \cdot (\overleftrightarrow{\Lambda} - \overleftrightarrow{L}) \cdot \overleftrightarrow{L}^{-1} \cdot \overleftrightarrow{M}_{pw} \cdot \tag{6}$$

A wall mode is unstable if the circuit equation $d\Phi_w/dt + \overleftrightarrow{R}_w \cdot \vec{I}_w = \vec{V}_w$ has growing solutions when the voltages applied to the external circuits, $\vec{V}_w$, are zero. The resistance matrix for the wall circuits, $\overleftrightarrow{R}_w$, is a positive matrix. When the inductance in the presence of a plasma, $\overleftrightarrow{\Lambda}$, is Hermitian, all of the matrices that have been introduced are Hermitian, and an unstable wall mode can arise only when $\overleftrightarrow{\Lambda}$ has a negative eigenvalue. Feedback systems [1–5] stabilize wall modes by eliminating growing solutions to the circuit equation by controlling the loop voltage $\vec{V}_w$.

The dimensionless matrix that characterizes the effect of the plasma on externally produced magnetic fields is the stability matrix $\overleftrightarrow{S}$. This matrix is defined as $\overleftrightarrow{S} \equiv \overleftrightarrow{L}^{1/2} \cdot \overleftrightarrow{\Lambda}^{-1} \cdot \overleftrightarrow{L}^{1/2}$. The square root of a positive, Hermitian matrix, such as $\overleftrightarrow{L}$, is defined using its eigenvectors and the square root of its eigenvalues. The stability matrix is Hermitian when the plasma obeys the constraints of ideal MHD and has the stability coefficients $-s$ as its eigenvalues. An eigenmode of the stability matrix is an unstable wall mode if its eigenvalue s is positive. Plasma rotation relative to the perturbation modifies the eigenvalues of $\overleftrightarrow{S}$ in two ways [5]: (i) the eigenvalues become complex numbers $-s + i\alpha$; (ii) the stability coefficient $-s$ is modified by slow rotation in a stabilizing sense in the form $s = s_0 - |\omega/\omega_0|^\sigma$, where ω is the angular frequency of the mode in the toroidal direction relative to the plasma, and ω_0 is a frequency that is of the order of the Alfvén frequency, v_A/qR. The coefficient σ is unity in a simple cylindrical plasma model.

Error field amplification arises if a flux component on the plasma surface is larger than the externally produced flux. The externally produced flux on the plasma surface can be written as $\vec{\Phi}_x = \overleftrightarrow{L} \cdot \vec{J}$, and the total flux can be written as $\vec{\Phi} = \overleftrightarrow{\Lambda} \cdot \vec{J}$. Therefore, $\vec{\Phi} = \overleftrightarrow{\Lambda} \cdot \overleftrightarrow{L}^{-1} \cdot \vec{\Phi}_x$, which can also be written as

$$\vec{\Phi} = \overleftrightarrow{L}^{1/2} \cdot \overleftrightarrow{S}^{-1} \cdot \overleftrightarrow{L}^{-1/2} \cdot \vec{\Phi}_x. \tag{7}$$

The eigenvalues of the matrix $\overleftrightarrow{S}^{-1}$, which can be written as $-(s + i\alpha)/(s^2 + \alpha^2)$, determine the extent of error field amplification.

IV. INDUCTANCE CALCULATIONS

The calculation of the plasma inductance $\overleftrightarrow{\Lambda}$ requires a method for finding the response of the plasma to an external perturbation. An external perturbation displaces the plasma surface by a distance $\vec{\xi}$, which is related to the normal field at the location of the unperturbed plasma surface by $\delta\vec{B} \cdot \vec{\nabla} r = \vec{B}_0 \cdot \vec{\nabla}(\vec{\xi} \cdot \vec{\nabla} r)$, with $\vec{B}_0$ the unperturbed magnetic field. The plasma response is the perturbation in the tangential magnetic field on the plasma surface that results from the imposition of a displacement of the surface. When the plasma obeys the constraints of ideal MHD, the calculation is relatively straight forward [5], and $\overleftrightarrow{\Lambda}$ is a Hermitian matrix.

When the plasma obeys the constraints of ideal MHD, the relation between the normal field on the plasma surface and the energy that is required to produce that normal field using a surface current on the plasma surface is $\delta W = \frac{1}{2}\vec{\Phi}^\dagger \cdot \overleftrightarrow{\Lambda}^{-1} \cdot \vec{\Phi}$, where δW is the standard ideal MHD expression [5]. The ideal MHD stability code DCON [8], which can find the relation between δW and the normal magnetic field on the plasma surface can be used to find $\overleftrightarrow{\Lambda}$. We have constructed a code that does this.

The calculation of the inductance of the plasma surface without the plasma, $\overleftrightarrow{L}$, can also be carried out using the DCON code. This is because the relation between a magnetic perturbation and the displacement of the field lines is given by $\delta\vec{B} = \vec{\nabla} \times (\vec{\xi} \times \vec{B}_0)$ in a toroidal system except on the resonant rational surfaces for the perturbation or when the perturbed toroidal loop voltage is non-zero. This follows from the mathematical statement that any vector can be represented as $\delta\vec{A} = \vec{\xi} \times \vec{B}_0 + \vec{\nabla} g$ if $\vec{B}_0 \cdot \vec{\nabla} g = \vec{B}_0 \cdot \delta\vec{A}$ has a well-behaved solution g. If one uses a pressureless tokamak equilibrium with very large value of the safety factor q and with the bounding magnetic surface having the shape $\vec{x}_s(\theta, \varphi)$, then the δW calculated using DCON will be same as for a vacuum magnetic field, and the inductance matrix will be $\overleftrightarrow{L}$.

The inductance of the plasma surface when no plasma is present, $\overleftrightarrow{L}$, is always Hermitian, and its eigenvalues increase in proportion to the typical poloidal mode number, m, in the perturbation. The inductance of the plasma surface when a plasma is present, $\overleftrightarrow{\Lambda}$, is Hermitian if the plasma obeys the constraints of ideal MHD and its eigenvalues are not only proportional to m, but are also inversely proportional to $-s$. The plasma stability parameter $-s$ is defined as the ratio of the energy required to produce a given magnetic

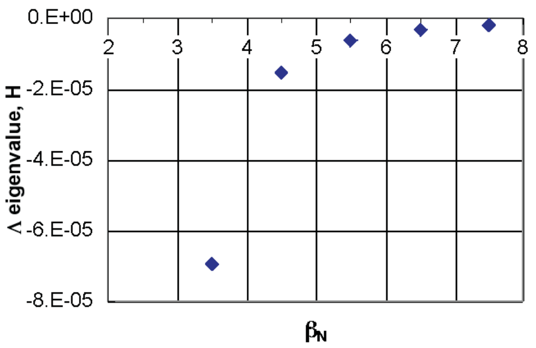

FIG. 1: Eigenvalues of the plasma inductance, $\overleftrightarrow{\Lambda}$, as a function of the normalized plasma pressure, β_N, for JT-60 tokamak.

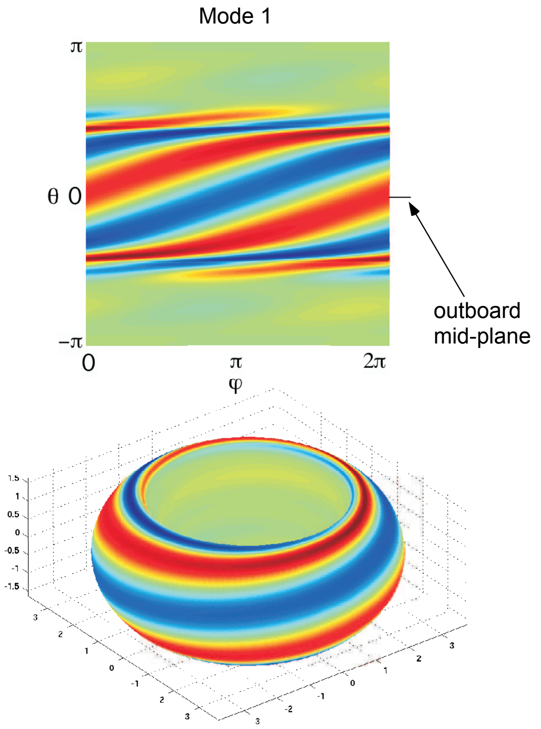

FIG. 2: Structure of the wall mode ($\beta_N = 2.5$) that becomes unstable when plasma pressure is increased to $\beta_N = 7.5$.

perturbation with versus without the plasma. As the plasma passes from being wall-mode stable to wall-mode unstable, an eigenvalue of $\overleftrightarrow{\Lambda}$ passes from positive to negative by going through infinity. An eigenvalue of $\overleftrightarrow{\Lambda}$ going to zero implies the plasma is acting as if it were a perfect rigid conductor for that mode.

Figure 1 shows eigenvalues of the $\overleftrightarrow{\Lambda}$ matrix for the wall-unstable mode as a function of the normalized plasma pressure, β_N, for the JT-60 tokamak. Normalized plasma pressure, β_N, is related to the plasma stability parameter, $-s$. As the plasma goes through marginal stability ($\beta \geq 3.5$), the eigenvalue of the unstable mode, λ, becomes negative. As β_N is increased to 7.5, λ asymptotically approaches zero. Figure 2 shows the structure of the wall-unstable mode, which essentially remains unchanged during the β_N-scan.

Acknowledgments

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