Ion temperature gradient modes in the presence of a sheared ion flow

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Abstract

Recently, a theory of spontaneous momentum generation due to the toroidally asymmetric excitation of ion temperature gradient (ITG) modes in the presence of a radial profile of toroidal ion velocity has been formulated [1]. We present a systematic study of the dispersion equation relevant to the electrostatic slab modes driven by an ITG perpendicular to a uniform magnetic field, in the presence of an inhomogeneous equilibrium fluid velocity of the ions.

Introduction

Sheared electric drift poloidal rotation in magnetically confined plasmas seems to play a major role in the suppression of radial energy leakage, thus inducing enhanced confinement regimes of operation [2]. The pattern of the electric drift flow is strictly related to the general problem of spontaneous momentum generation and of its transport within a magnetized plasma in axisymmetric toroidal plasmas, which is also in a close relationship with the physics of accretion disks. Toroidal plasma rotation is frequently characterizing auxiliary heated plasmas [3], while poloidal flows can be produced by IBW [4]. Recently a theory of spontaneous momentum generation due to the toroidally asymmetric excitation of radially localized ion temperature gradient (ITG) modes in the presence of a radial profile of toroidal ion velocity has been formulated [1]. The resulting radial transport of the angular momentum during external plasma heating is in qualitative agreement with the plasma dynamics observed in Alcator C-MOD [3]. In this work we present a detailed investigation of the relevant dispersion relation derived in the frame of the slab two-fluid drift model [5], where equilibrium inhomogeneous ion velocities are introduced, \( \mathbf{U}_\parallel(x)\hat{\mathbf{z}} \) parallel to the ambient magnetic field, \( \mathbf{B}_0 = B_0\hat{\mathbf{z}} \), and \( \mathbf{U}_\perp(x)\hat{\mathbf{y}} \) perpendicular to both \( \mathbf{B}_0 \) and to the unperturbed density and temperature gradients, \( \nabla \parallel \hat{\mathbf{x}} \). The objective is to investigate the role of the asymmetry, introduced by the sheared flows, in the excitation of waves with opposite values of \( k_\parallel/k_\perp \), capable of affecting the anomalous transport of angular momentum.
The electrostatic dispersion relation

The linearized two-fluid equations in the drift approximation, in a plasma slab non uniform in the $x$ direction, where the electrostatic potential perturbation is assumed to be localized around $x=x_0$, are written as:

\[ \omega \tilde{n}_k = n k y \tilde{v}_{l,k} - \frac{c}{B_0} n' k_z \tilde{\varphi}_k \]  
(1)

\[ M n \tilde{v}_{l,k} = q n k\left(1 - \frac{k_y U'_y}{k_\parallel \Omega} \right) \tilde{\varphi}_k + k_\parallel \tilde{p}_k \]  
(2)

\[ \omega \tilde{p}_k = -\frac{c}{B_0} p' k_z \tilde{\varphi}_k + \gamma p k_\parallel \tilde{v}_{l,k} \]  
(3)

\[ \frac{n}{T_c} = \frac{c \tilde{q}_k}{A} \]  
(4)

\[ n = n_i = \frac{n_s}{Z} \]  
(5)

Here, the 2D space Fourier transform

\[ \tilde{A}(x,y,z,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{A}_k(x,t)e^{i(k_x x + k_y y + k_z z)} \]  
(6)

has been introduced, where the time dependence of each Fourier coefficient is in the form of $\tilde{A}_k(t) = \tilde{A}_k(t)e^{-i\omega_k t}$, with the complex frequency $\omega_k = \omega_k^R + i \omega_k^I$, $\omega_e = \omega_k^R$, and $\tilde{A}_k(x) = \tilde{A}_k^c(x)$. Moreover, in Eqs.(1-4) the $x$-dependent frequency $\omega(x) = \omega_k^R - k_y U'_y(x) - k_z U'_z(x)$, the perpendicular $k_\perp = k_y$ and the parallel $k_\parallel = k_z$ components of the wave-vector, have been defined. Finally, $n_i(x), T_e(x), p_a(x) = n_i(x) T_e(x)$ are the density, temperature and pressure of the $a$-species, respectively, and $\gamma$ is the adiabatic index of the ions. The apex indicates the total differentiation with respect to $x$. From Eqs.(1-5) the dispersion relation follows [1]

\[ 1 = \frac{k_i}{\omega} \frac{v_{se}}{\omega} - D_B (\frac{k_i}{\omega})^2 (U'_y + \frac{k_i}{\omega} p'_i - \frac{k_z}{k_\parallel} \Omega) \left(1 \right. \left. \omega - \frac{k^2_\perp v^2}{\omega^2} \right) \]  
(7)

where $v_{se}(x) = -D_B n', D_B = c T_e / e B_0$, $\rho_i(x) = n_i(x) M_e$, $\Omega = q_i B_0 / M_e$, $v_i = (T_i / M_i)^{1/2}$, $q_i = Ze$. Eq.(7) gives a cubic equation in $\omega$, which can be put in the approximate form

\[ \omega^3 - \gamma k_i c^2 \left(1 - \frac{Z}{\gamma} \frac{k_i U'_y}{\Omega} \right) \omega + Z k_i c^2 \omega_{se} = 0 \]  
(8)

for $\eta_i = (\ln T_i)' / (\ln n_i)' > 1, \tau = T_i / T_e > 1, \frac{\omega_{se}}{\omega} << 1$. In Eq.(8), $\omega_{se}(x) = k_i c T_i / (Ze b_0)$. By neglecting the $U'_y$-term in Eq.(8), we recover the dispersion equation considered by Coppi et
al. in 1967 [5]. From inspection of Eqs.(7,8) it is seen that a non-zero value of $k_\parallel$ introduces the effects of the inhomogeneous ion population, through its pressure and parallel velocity profiles, which can stabilize or destabilize the system. Moreover, and most important to our scopes, the $U_i$-term introduces an odd dependence on $k_\parallel$. According to the standard notations [6], there is an unstable solution of Eq.(8) if

$$\Delta = q^3 + r^2 = \frac{Z^2\tau^2}{4} (k_\parallel c_1 c_s)^3 \rho_s^2 \left[ k_{T_i} - \left( \kappa_{T_i} \right)^2 \right] > 0,$$

where $\kappa_{T_i} = -(\ln T_i)'$, and

$$\kappa_{T_i} \equiv \kappa_{T_i}^c = \frac{2}{\gamma^{3/2} Z} \frac{U_i}{\Omega_c} \frac{k_\parallel \Omega_c}{\rho_s} \left( \kappa_{T_i} \right)^1 \left( \kappa_{T_i} \right)^3$$

is a “critical” ion temperature gradient; $c_s = (T_s)^{1/2}$, $\rho_s = c_s/\Omega_c$ and $\kappa_{T_i}^c = \frac{2}{\gamma^{3/2} Z} \frac{U_i}{\Omega_c} \frac{k_\parallel \Omega_c}{\rho_s}$ refers to the case where no parallel velocity shear is present.

Note that $\kappa_{T_i}$ can be made equal to zero with a suitable choice of $U_i$. For $|1 - \left( \kappa_{T_i}^c/\kappa_{T_i} \right)^2| << 1$, the unstable mode has $\omega_k = \left( \frac{Z}{2} \right)^{1/3} (k_\parallel c_s)^{2/3} \omega_{T_i}^{1/3} \approx \left( \frac{\gamma}{3} \right)^{1/2} \text{sign}(k_\parallel) |k_\parallel| v_{ti} \left( 1 - \frac{Z}{\gamma} \left( \frac{k_\parallel}{\Omega_c} \right)^{1/2} \right)$ and

$$\gamma_k \approx \frac{Z^{1/3} \gamma^{1/6}}{3^{1/2}} \left( k_\parallel c_s \right)^{2/3} \omega_{T_i}^{1/3} \frac{\left( k_{T_i} \right)^{1/2}}{\left( \kappa_{T_i} \right)^{1/2}} - 1.$$ 

**Consequences of $k_\parallel/k_\perp$ symmetry breaking**

Let us consider a plasma region around $x = x_0$ where $U_i' > 0$. Let’s choose two modes with the same $k_\perp > 0$ and opposite $k_\parallel$, that is, $k_{\parallel 1} = -k_\parallel < 0$ and $k_{\parallel 2} = k_\parallel > 0$. Assume that at $x = x_0$ mode 1 is marginally stable, that is $\gamma_{k_1} = 0$, due to the fact that locally $\kappa_{T_i} \equiv \left( \kappa_{T_i}^c \right)_1$. We can use $\left( \kappa_{T_i}^c \right)_1$ to calculate $\Delta_2$ relevant to mode 2; it takes the form

$$\Delta_2 = \frac{2Z}{3} k_\perp k_\parallel c_s^6 \frac{U_i}{\Omega_c} \left[ 3\gamma^2 \tau^2 + \frac{Z^2}{k_\perp^2} (k_\parallel c_s)^{2/3} \left( \frac{U_i}{\Omega_c} \right)^2 \right] > 0,$

corresponding to an unstable solution. For small values of the sheared parallel ion velocity, that is $\frac{k_\parallel}{k_\perp} \frac{U_i}{\Omega_c} << \frac{\gamma\tau}{Z}$, we get

$$\omega_k^R \approx -\left( \frac{\gamma}{3} \right)^{1/2} \frac{\left( k_\parallel c_s \right)^{2/3}}{\omega_{T_i}^{1/3}} \text{ and } \gamma_k \approx \left( \frac{2Z}{3} \right)^{1/2} \left( k_\parallel c_s \right)^{2/3} \left( \frac{U_i}{\Omega_c} \right)^{1/2},$$

thus being

$$\frac{\gamma_{k_2}}{\omega_{k_2}^R} << 1.$$

In the opposite limit, for $\frac{k_\parallel}{k_\perp} \frac{U_i}{\Omega_c} >> \frac{\gamma\tau}{Z}$, putting $\sigma_0 = \left( \left( 2^{1/2} + 1 \right)^{1/3} \pm \left( 2^{1/2} - 1 \right)^{1/3} \right)/2,$
A quasilinear flux of parallel ion velocity in the $x$-direction is associated with waves propagating in the $(y,z)$ plane. For a spectrum of growing modes, i.e. with $\gamma_k > 0$ (as it is the case for wave 2), and in the limit $k^2 U''_i / |\sigma_k|^2 << 1$, the flux takes the form [1]

$$\Gamma_{x,v_{\parallel}} = \rho_1 \langle \bar{v}_{Ex} \bar{v}_{ii} \rangle = -\frac{\rho_1}{2\pi B_0} \int_{-\infty}^{\infty} \, dk_{\perp} \gamma_k c_2 k^2 |\sigma_k|^2 \frac{k_{\parallel}^2}{|\sigma_k|^2} \bar{v}_k \frac{e^{2\gamma i}}{2} \left( U''_i + 2 \frac{k_{\parallel}^2}{|\sigma_k|^2} \frac{p_i'}{k_{\parallel}} - \frac{\bar{k}_i}{k_{\parallel}} \frac{k_{\perp}}{\Omega_c} \right)$$  \hspace{1cm} (9)

For $U''_i = 0$, waves with positive and negative $k_{\parallel}$ are likely to be excited with the same growth rate. In this situation, Eq.(9) shows that there is a balance between quasilinear momentum fluxes, inward and outward, brought out by different components of the wave spectrum, and no net momentum transport in $x$ takes place. On the contrary, if for example $U''_i > 0$, the situation can occur in which only modes with $k_{\parallel} > 0$ are excited, (that is, waves of type 2). Then the integrand in Eq.(9) is no longer an odd function of $k_{\parallel}$, the radial ($x$) flux of parallel momentum associated with waves having $k_{\parallel} / k_{\perp} < 0$ (definitely inward for $p_i' < 0$ and $U''_i = 0$) is suppressed, while that produced by waves with $k_{\parallel} / k_{\perp} > 0$ is non-zero. Momentum flux can be made inward for sufficiently large values of $U''_i > 0$. Indeed, for $|\bar{v}_k |^2 = \delta(k_{\perp} - k_{\parallel}) \left[ I_1 \delta(k_{\parallel} + \bar{k}_i) + I_2 \delta(k_{\parallel} - \bar{k}_i) \right]$, with $\gamma_{-v_{\parallel},0} = 0$, Eq.(9) becomes

$$\Gamma_{x,v_{\parallel}} = -\frac{cZ \epsilon_0 k_{\parallel}^2}{2\pi B_0} \frac{\gamma_k}{|\sigma_k|^2} e^{2\gamma i} \left[ 4 \frac{U''_i}{\Omega_c} - \frac{4 \gamma / \sqrt{3} / \sqrt{3}}{Z} \frac{k_{\parallel}^2 c_3^2}{|\sigma_k|^2} \frac{\sigma_{R}^{2}}{Z} \left( 1 + \frac{Z}{\gamma \tau} \frac{k_{\parallel} \Omega_c}{k_{\parallel}} \right) \frac{k_{\parallel}^2}{|\sigma_k|^2} \right]$$  \hspace{1cm} (10)

where the index “k” means $(\bar{k}_i,k_{\parallel})$. Considering the minus sign and the square brackets only in Eq.(10), we obtain that for $(k_{0}/|\bar{k}_i|)U''_i / \Omega_c << \gamma \tau / Z$, $\Gamma_{x,v_{\parallel}} \propto \left[ 4 \gamma \tau / (3Z) - 1 \right] |\bar{k}_i| / k_{\parallel} > 0$, and for $(k_{0}/|\bar{k}_i|)U''_i / \Omega_c >> \gamma \tau / Z$, $\Gamma_{x,v_{\parallel}} \propto -U''_i / \Omega_c < 0$.

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