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Lagrangian formulation of electrostatic plasma waves: Application to dust-acoustic waves

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Abstract

A generic methodology is proposed for the description of electrostatic plasma modes via a general collisionless (unmagnetized) Lagrangian (single-) fluid model. The linear and weakly nonlinear oscillations regimes are investigated. The modulational stability profile of dust acoustic waves is discussed, as a case study.

1. Introduction. A variety of known electrostatic plasma waves [1] refer to propagating oscillations of one dynamical plasma constituent α (mass m_α, charge $q_\alpha \equiv s_\alpha Z_\alpha e$; e is the absolute electron charge; $s_\alpha = q_\alpha/|q_\alpha| = \pm 1$ is the charge *sign*) against one (or more) background constituent(s) α' (mass $m_{\alpha'}$, charge $q_{\alpha'} \equiv s_{\alpha'} Z_{\alpha'} e$, similarly), which is (are) considered to obey a known distribution, e.g. in a fixed (uniform) or in a thermalized (Maxwellian) state, depending on the particular aspects (e.g. frequency scales) of the physical system considered. For instance, the *ion-acoustic* mode [1] refers to ions ($\alpha = i$) oscillating against a Maxwellian electron background ($\alpha' = e$), the *dust-acoustic* mode [2] refers to low-frequency oscillations of charged dust grains ($\alpha = d$) against a thermalized ion and electron background ($\alpha' = i, e$), and so forth. An electrostatic plasma mode is thus often efficiently described via a single fluid model, where the dynamics of the inertial species α obeys the density n_α (conservation) and the mean fluid velocity $\mathbf{u}_\alpha$ (moment evolution) equations, considering the electric force $\mathbf{F} = -q_\alpha \nabla\phi$ in the latter (and omitting the Lorentz force therein, since propagation along the external magnetic field lines, $\sim \hat{z}$, is considered). The pressure dynamics (i.e. the temperature effect) is omitted in this (cold fluid) description. The system is then closed by considering Poisson's equation, for the electric potential ϕ. Overall neutrality is assumed at equilibrium, i.e. $\sum_{\sigma=\alpha,\{\alpha'\}} n_{\sigma,0}\, q_\sigma = 0$.

2. The model. In this brief paper, we aim in suggesting a *generic* model for the study of the modulational (in)stability of electrostatic plasma waves, by employing a Lagrangian description, i.e. looking into a moving frame which follows the fluid motion at velocity $\mathbf{u}_\alpha$. This is achieved by introducing the Lagrangian variables $\{\xi, \tau\}$,

$$\xi \;=\; z \,-\, \int_0^\tau u(\xi, \tau')\, d\tau'\,, \qquad\qquad \tau = t\,. \tag{1}$$

This Lagrange transformation has been widely studied in fluid mechanics and in plasma physics [3 - 5]. See that the two sets of coordinates coincide at $t = 0$. The space and time gradients are now $\partial/\partial z \to \beta^{-1}\,\partial/\partial\xi$ and $\partial/\partial t \to \partial/\partial\tau - \beta^{-1}\,u_\alpha\,\partial/\partial\xi$, where

$$\beta(\xi,\tau) \equiv \frac{\partial x}{\partial \xi} = 1 + \int_0^\tau d\tau' \frac{\partial}{\partial\xi} u_\alpha(\xi,\tau')\,. \tag{2}$$

Note that the convective derivative $D \equiv \partial/\partial t + u\,\partial/\partial x$ is now plainly identified to $\partial/\partial\tau$. Also notice that β satisfies $\beta(\xi,\tau=0)=0$, and $\partial\beta(\xi,\tau)/\partial\tau = \partial u_\alpha(\xi,\tau)/\partial\xi$.

The Lagrangian transformation defined above leads to a new set of reduced equations

$$n_\alpha(\xi,\tau) = \beta^{-1}(\xi,\tau)\,n_\alpha(\xi,0) \tag{3}$$

$$\frac{\partial u_\alpha(\xi,\tau)}{\partial\tau} = s_\alpha \frac{Z_\alpha e}{m_\alpha} E(\xi,\tau) \equiv -s_\alpha \frac{Z_\alpha e}{m_\alpha}\,\beta^{-1}(\xi,\tau)\,\frac{\partial\phi(\xi,\tau)}{\partial\xi}\,, \tag{4}$$

$$\beta^{-1}(\xi,\tau)\,\frac{\partial E(\xi,\tau)}{\partial\xi} = 4\pi\,s_\alpha\,Z_\alpha\,e\,[n_\alpha(\xi,\tau) - \hat{n}\,n_{\alpha,0}]\,, \tag{5}$$

$$\left(\frac{\partial}{\partial\tau} - \beta^{-1}u_\alpha\,\frac{\partial}{\partial\xi}\right)E(\xi,\tau) = -4\pi\,s_\alpha\,Z_\alpha\,e\,n_\alpha(\xi,\tau)\,u_\alpha(\xi,\tau)\,. \tag{6}$$

The (dimensionless) quantity $\hat{n} \equiv -\sum_{\alpha'} n_{\alpha'}q_{\alpha'}/(n_{\alpha,0}q_\alpha)$ is in fact a known exact function of ϕ, to be defined for each specific problem under consideration. Poisson's equation is obtained by combining Eqs. (4) and (5):

$$\beta^{-1}\,\frac{\partial}{\partial\xi}\left(\beta^{-1}\,\frac{\partial\phi}{\partial\xi}\right) = -4\pi\,s_\alpha\,Z_\alpha\,e\,(n_\alpha - \hat{n}\,n_{\alpha,0})\,. \tag{7}$$

In principle, one's aim is to solve the system of Eqs. (3) to (6) or, in terms of ϕ, Eqs. (3), (4) and (7), for a given initial condition $n_\alpha(\xi,\tau=0) = n_0(\xi)$, and then invert back to $\{z,t\}$. However, this abstract scheme is definitely not a trivial task to accomplish.

3. Nonlinear electrostatic oscillations. Combining Eqs. (4) to (6), one obtains

$$\frac{\partial^2 u_\alpha}{\partial\tau^2} = -\omega_{p,\alpha}^2\,\hat{n}\,u_\alpha\,, \tag{8}$$

where $\omega_{p,\alpha}$ is the plasma frequency $\omega_{p,\alpha} = (4\pi n_{\alpha,0}Z_\alpha^2 e^2/m_\alpha)^{1/2}$. Despite its apparent simplicity, Eq. (8) is *neither* an ordinary differential equation *nor* a closed evolution equation for the mean velocity $u_\alpha(\xi,\tau)$: recall that the (normalized) background particle density $\hat{n}$ depends on the potential ϕ, whose evolution in turn involves $u_\alpha(\xi,\tau)$ [via $\beta(\xi,\tau)$] and $n_\alpha(\xi,\tau)$. Eq. (8) suggests that the system performs nonlinear oscillations at a frequency $\omega = \omega_{p,\alpha}\,\hat{n}^{1/2}$. Near equilibrium $\hat{n} \approx 1$ (thanks to charge quasi-neutrality) and one plainly recovers a linear oscillation at the plasma frequency $\omega_{p,\alpha}$. Unfortunately this apparent simplicity, which might *in principle* enable one to solve for $u(\xi,\tau)$ and then obtain $\{\xi, \tau\}$ in terms of $\{z, t\}$ and vice versa (cf. [3]) is in general *illusory*: the oscillations described by Eq. (8) are intrinsically *nonlinear* and (unless $\hat{n} = const.$; cf. e.g. Refs. [3, 4] for electron-acoustic waves), one has to retain all of Eqs. (3) to (7).

4. Reductive perturbation analysis. The above system of evolution equations describes electrostatic oscillations in the form of harmonic waves, i.e. $\mathbf{S} = \mathbf{S_0} \exp[i(k\xi - \omega\tau)] + \text{c.c.}$ ($\mathbf{S}$ denotes the state vector $\{n_\alpha, u_\alpha, E, \phi, \beta\}$). In order to study the modulational stability profile of these electrostatic waves and model the nonlinear harmonic generation mechanism entering into play when their amplitude becomes non-negligible, we consider small deviations from the equilibrium state $\mathbf{S}^{(0)} = (1, 0, 0, 0, 1)^T$, i.e. $\mathbf{S} = \mathbf{S}^{(0)} + \epsilon \mathbf{S}^{(1)} + \epsilon^2 \mathbf{S}^{(2)} + ...$, where $\epsilon \ll 1$ is a smallness parameter. We have assumed that $Sj^{(n)} = \sum_{l=-\infty}^{\infty} S_{j,l}^{(n)}(Z, T)\, e^{il(k\xi - \omega\tau)}$ (for $j = 1, 2, ..., 5$; $S_{j,-l}^{(n)} = S_{j,l}^{(n)\,*}$, for reality), thus allowing the wave amplitude to depend on the stretched (*slow*) Lagrangian coordinates $Z = \epsilon(\xi - v_g\, \tau)\,, \quad T = \epsilon^2\, \tau$ [where $v_g = \omega'(k)$ is the wave group velocity]. For convenience, time and space are scaled by (appropriately chosen, for a given problem) scales $\hat{\tau}$ (e.g. $\omega_{p,\alpha}^{-1}$) and $\hat{\xi} = V\tau$ [where $V = (k_B T_* / m_\alpha)^{1/2}$; T_* is a characteristic temperature, to be appropriately defined] also n, u and ϕ are scaled over $n_{\alpha,0}$, V and $k_B T_* / |q_\alpha|$, respectively. Taylor expanding $\hat{n}(\phi)$ near $\phi \approx 0$ (viz. $\phi \approx \epsilon\phi_1 + \epsilon^2\phi_2 + ...$),

$$\begin{aligned} \hat{n} &\approx 1 + c_1\phi + c_2\phi^2 + c_3\phi^3 + ... \\ &= 1 + \epsilon c_1 \phi_1 + \epsilon^2 (c_1\phi_2 + c_2\phi_1^2) + \epsilon^3 (c_1\phi_3 + 2c_2\phi_1\phi_2 + c_3\phi_1^3) + \ldots . \end{aligned} \tag{9}$$

The coefficients c_j ($j = 1, 2, ...$), to be determined from (the exact definition of) $\hat{n}$ for any given problem, contain all the essential dependence on the plasma parameters.

One is now left with the task of isolating orders in ϵ^n (i.e. $n = 1, 2, ...$) and successively solve for the harmonic amplitudes $S_{j,l}^{(n)}$. The equations for $n = l = 1$ yield

$$\phi_1^{(1)} = \psi\,, \qquad n_1^{(1)} = -\alpha_1^{(1)} = s(k^2/\omega^2)\psi\,, \qquad u_1^{(1)} = s(k/\omega)\psi\,, \qquad E_1^{(1)} = -i\, k\psi\,, \tag{10}$$

along with the dispersion relation $\omega^2 = k^2/(k^2 + sc_1)$ †. For $n = 2$, we obtain the amplitudes of the 2nd harmonics $\mathbf{S}_2^{(2)}$ and constant (*'direct current'*) terms $\mathbf{S}_0^{(2)}$, as well as a finite contribution $\mathbf{S}_1^{(2)}$ to the first harmonics (expressions omitted here, for brevity).

To order $\sim \epsilon^3$, the equations for $l = 1$ yield an explicit compatibility condition in the form of a nonlinear Schrödinger–type equation

$$i\, \frac{\partial \psi}{\partial T} + P\, \frac{\partial^2 \psi}{\partial Z^2} + Q\, |\psi|^2\, \psi = 0\,. \tag{11}$$

The *dispersion coefficient* P is related to the curvature of the dispersion curve as $P = \omega''(k)/2 = -3\omega^3(1-\omega^2)/(2k^2) < 0$ ($\forall\ k$). The *nonlinearity coefficient* Q, due to carrier wave self-interaction, is given by the expression

$$Q = +\frac{\omega^3}{12\,k^4}\Big\{(-9c_1^4 + 12c_1^2c_2 - 4c_2^2) + [6(s-3)c_1c_2 - 15sc_1^3 + 18sc_3]\,k^2 \\ -[3c_1^2 + 6(3s+1)c_2]\,k^4 + 3sc_1\,k^6\Big\}\,. \qquad (12)$$

At long wavelengths, Q behaves as $Q \approx -(3c_1^2 - 2c_2)^2/(12k)$ (for any charge sign $s = \pm 1$). We note a strong modification in the wave's nonlinear profile (cf. the form of the Q coefficient) in this Lagrangian description, with respect to its Eulerian counterpart.

5. A case study: Dust Acoustic waves. Applying the above formalism to dust acoustic waves [2, 7], viz. $\alpha = d$ (dust) and $\alpha' = e, i$, with $n_e = n_{e,0}\exp[e\phi/(k_BT_e)]$ and $n_i = n_{i,0}\exp[-Z_ie\phi/(k_BT_i)]$ (and $s = -1/+1$ for negatively/positively charged dust grains), one obtains: $c_1 = -\theta_1(1+\mu\theta_2)/(1-\mu)$, $c_2 = +\theta_1^2(1-\mu\theta_2^2)/[2(1-\mu)]$, and $c_3 = -\theta_1(1-\mu\theta_2^3)/[6(1-\mu)]$, having defined the dust parameter $\mu = n_{e,0}/(Z_in_{i,0})$, the parameters $\theta_1 = Z_iT_{eff}/(Z_dT_i)$ and $\theta_2 = T_i/Z_iT_e$, the scales $\hat{\tau} = \omega_{p,d}^{-1}$ and $\hat{\xi} = c_d\hat{\tau}$, where $c_d = \omega_{p,d}(\lambda_{D,e}^{-2} + \lambda_{D,i}^{-2})^{-1/2} \equiv (T_{eff}/m_d)^{1/2}$, and making use of the neutrality condition at equilibrium. Note that the sign of $c_1/c_2/c_3$ is $s/-s/s$, respectively (so $sc_1 > 0$). A simple numerical analysis shows that long wavelength dust-acoustic waves will be modulationally unstable to perturbations, since $PQ > 0$, which may result in the formation of *bright-type* envelope excitations, while wavelengths shorter than a threshold will be stable, and may propagate as *dark/gray-type* envelope modulated wavepackets [6, 7].

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† The same dispersion relation is obtained (upon recovering dimensions) from the initial (Eulerian) system of hydrodynamic equations, as physically expected [6].

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