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The structure of fields with electric and magnetic components parallel to each other

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Chu et al.[1] showed sometime ago the existence of transverse electromagnetic fields with electric and magnetic fields parallel to one another, whilst some interesting clarifications on the mathematical aspects have been presented by Gray [2], who considered them as a singular solution of the Maxwell set of equations.

It can be shown directly that Arnold's solution [3], given first in the context of Beltrami fields,

$$A_x = (A\sin kz + C\cos ky)\cos\omega t, \qquad A_y = (B\sin kx + A\cos kz)\cos\omega t,$$

$$A_z = (C\sin ky + B\cos kx)\cos\omega t \qquad \mathbf{A} = (A_x, A_y, A_z), \tag{1}$$

can be considered as a more general form of a vector potential for the derivation of electromagnetic fields with electric and magnetic fields parallel to one another. This solution represents six waves with a dispersion relation given by $\omega^2 = c^2k^2$ and eight null points. It is noted that such a field admits of the first integral of energy recently found [4]. The set of equations given in [1] can be retrieved upon setting B = C = 0 in (1).

However, Landau & Lifshitz [4] suggested that given the relative meaning of these two fields, systems of reference can be found, where the two fields are parallel to each other and the velocity of those frames should be given by the condition $\mathbf{E}' \wedge \mathbf{B}' = 0$. These latter fields are those observed in the moving frame of reference. Then the velocity is defined by

$$\left(\frac{\mathbf{v}}{c}\right)^2 - \frac{|\mathbf{E}|^2 + |\mathbf{B}|^2}{|\mathbf{E} \wedge \mathbf{B}|}\left(\frac{\mathbf{v}}{c}\right) + 1 = 0 \tag{2}$$

which admits the solutions

$$\frac{v}{c} = \frac{1}{2}\frac{1}{|\mathbf{E} \wedge \mathbf{B}|}\left[E^2 + B^2 \pm \sqrt{(E^2 - B^2)^2 + 4(\mathbf{E}\cdot\mathbf{B})^2}\,\right] \tag{3}$$

E and **B** are the electric and magnetic fields in the system at rest and perpendicular to each other. It is noted that the expressions under the sign of the square root are the invariants of the electromagnetic field. Since for an observer in the rest frame of reference **E** is perpendicular to **B** the two roots are found to be

$$\frac{v}{c} = \frac{E}{B} \text{ and } \frac{v}{c} = \frac{B}{E} \tag{4}$$

The admissible solution has to be decided on the relative magnitude of the electric and magnetic field, so that the postulates of special relativity are not violated. In vectorial notation expressions (4) can be written as

$$\mathbf{v} = c\frac{\mathbf{E} \wedge \mathbf{B}}{B^2} \qquad \mathbf{v} = c\frac{\mathbf{E} \wedge \mathbf{B}}{E^2}$$

It is noted that for E = B the velocity of the moving frame of reference for the two fields to be parallel, should be that of the speed of light, i.e.,

$$v = c$$

It is a rather obvious conclusion that there is no outflow of energy from such systems.

With the proof of the existence electromagnetic systems with their fields parallel to one another the question arises naturally as to their actual expressions in different systems of coordinates. In the next section such expressions will be constructed analytically in a cylindrical system of coordinates.

The derivation

The Fourier analysed Maxwell's equations in the SI-system read

$$\nabla \wedge \mathbf{E} = i\,\omega\mu\mathbf{H} \qquad \nabla \wedge \mathbf{H} = -i\omega\varepsilon\mathbf{E} \tag{5}$$

$$\nabla \cdot \mathbf{E} = 0 \qquad \nabla \cdot \mathbf{H} = 0 \tag{6}$$

for an exp(- i ω t) time dependence. Similarly the spatial dependence will be assumed of the form exp[i(kz + m ϕ)]. The magnetic field is given by **B = μH,** μ= the magnetic permeability. For

$$\mathbf{E} = \alpha\mathbf{B} \tag{7}$$

the compatibility condition between (5), (6) shows that

$$\alpha = \frac{i}{\sqrt{\varepsilon\mu}} \tag{8}$$

The rotation of the fields (5) gives

$$i\sqrt{\frac{\mu}{\varepsilon}}\left(\frac{m}{r}H_z - k_z H_\phi\right) = \omega\mu H_r \qquad \frac{m}{r}H_z - k_z H_\phi = i\sqrt{\frac{\mu}{\varepsilon}} i\omega\varepsilon H_r \tag{9}$$

$$i\sqrt{\frac{\mu}{\varepsilon}}(ik_zH_r - \frac{\partial H_z}{\partial r}) = i\omega\mu\, H_\phi \qquad ik_zH_r - i\sqrt{\frac{\mu}{\varepsilon}}i\omega\varepsilon H_\phi \tag{10}$$

$$i\sqrt{\frac{\mu}{\varepsilon}}[\frac{1}{r}\frac{\partial}{\partial r}(rH_\phi) - i\frac{m}{r}H_r] = i\omega\mu H_z \qquad i\frac{\partial}{\partial r}(rH_\phi) - i\frac{m}{r}H_r = -i\sqrt{\frac{\mu}{\varepsilon}}i\omega\varepsilon H_z \tag{11}$$

and the divergence of the fields (6) combine into the following expression

$$\frac{1}{r}\frac{\partial}{\partial r}(rH_r) + i\frac{m}{r}H_\phi + ik_zH_z = 0 \tag{12}$$

Employing the abbreviation $\omega\mu\sqrt{\varepsilon/\mu} = k_0$ the system of equations

$$\frac{m}{r}H_z - k_zH_\phi = -ik_0H_r, \quad ik_zH_r - \frac{\partial H_z}{\partial r} = k_0H_\phi, \quad \frac{1}{r}\frac{\partial}{\partial r}(rH_\phi) - i\frac{m}{r}H_r = k_0H_z \tag{13}$$

obtains together with the compatibility condition

$$\frac{1}{r}\frac{\partial}{\partial r}(rH_r) + i\frac{m}{r}H_\phi + ik_zH_z = 0 \tag{14}$$

Eliminating the H_r, H_ϕ components the differential equation

$$\frac{d^2H_z}{dr^2} + \frac{1}{r}\frac{dH_z}{dr} + (k_c^2 - \frac{m^2}{r^2}H_z = 0 \tag{15}$$

is found, which can be also retrieved from substitution into (14). It is noted that

$$k_c^2 = k_0^2 - k_z^2, \text{ and } k_0^2 = \omega^2\mu\varepsilon \tag{16}$$

The solution finite at the origin is the Bessel function with argument $k_c r$, so that

$$H_r = \sum_{m-o}^{\infty} iH_{om}\frac{k_0}{k_c^2}(\frac{m}{r}J_m + \frac{k_ck_z}{k_o}J'_m) \tag{17}$$

$$H_\phi = -\sum_{m=o}^{\infty} H_{om}\frac{k_z}{k_c^2}(\frac{m}{r}J_m + \frac{k_ok_c}{k_z}J'_m) \tag{18}$$

$$H_z = \sum_{m=o}^{\infty} H_{om}J_m(k_cr) \tag{19}$$

As a further check it can be verified that a substitution of the values H_r, H_ϕ into $\nabla\cdot\mathbf{H} = 0$ reproduces equation (15).

We proceed now to calculate the time - averaged power transmitted by these modes along the $\mathbf{e}_z$-direction. The standard expression for power transmission (see for instance [6])

$$P_z = \mathrm{Re}\{\int(\mathbf{E}\wedge\mathbf{H}^*)\cdot d\mathbf{S}_z\} \tag{20}$$

gives

$$P_z = \mathrm{Re}\,\frac{1}{2}\int_0^{2\pi}\int_0^{a} \mathbf{E}\wedge \mathbf{H}^* \cdot \mathbf{e}_z r dr d\theta = \mathrm{Re}\{i\pi\sqrt{\frac{\mu}{\varepsilon}}\int_0^{a}(H_r H_\phi^* - H_\phi H_r^*))rdr\}$$

Employing then expressions (17) – (19) it is found

$$P_z = 2\pi\sqrt{\frac{\mu}{\varepsilon}}H_0^2\frac{k_0 k_z}{k_c^4}\int_0^{k_c a}[\frac{m^2}{x^2}J_m^2 + \frac{k_0^2 + k_z^2}{k_0 k_z}\frac{m}{x}J_m J'_m + J'^2_m]xdx \qquad (21)$$

$x = k_c r$.

Summary

The explicit expressions for the velocity of frames of reference in which the electric and magnetic fields are parallel to one another were found and the analytic expressions were constructed in cylindrical coordinates. The field energy cannot be radiated away, since the Poynting vector is zero, although it can be transported.

References

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