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Adiabatic Invariants of Energetic Particle Motion in a Stellarator

A.V.Tykhyy, Yu.V.Yakovenko

Institute for Nuclear Research, Prospekt Nauky 47, Kyiv 03680, Ukraine

1. Introduction. Stellarators are characterized by complex configurations of the magnetic field with numerous significant Fourier harmonics. The equations of the collisionless motion of the plasma particles in such configurations cannot be easily integrated. An efficient way to study the particle motion in this case is by using adiabatic invariants. In particular, the motion of the locally trapped particles is described by the well-known longitudinal adiabatic invariant $I_\parallel$. An adiabatic invariant for the locally passing particles was derived in Ref. [1]. These invariants are suitable for describing the bounce-averaged motion (or the motion averaged upon one field period for the passing particles), disregarding the particle drifts within the averaging interval. The aim of this work is to derive adiabatic invariants of the guiding center motion, which represent all particle drifts, including the drifts within one bounce. Such detailed knowledge of the particle motion is required, in particular, for calculations of the drive of Alfvén instabilities by energetic particles. We restrict ourselves to two cases, the particles ever remaining locally trapped and those ever remaining passing. The transitioning particles, i.e., the particles that transition from the locally trapped state to the locally passing state and vice versa, are not considered here.

2. Derivation method. We consider a magnetic configuration with nested flux surfaces and proceed from the guiding-center Lagrangian differential form [2]

$$\boldsymbol{\gamma} = \frac{e}{c}(X^\psi \mathrm{d}X^\theta - I\mathrm{d}X^\phi) + \frac{Mv_\parallel}{B}\,\mathbf{B}.\mathrm{d}\mathbf{X} - K\mathrm{d}t, \tag{1}$$

where $\mathbf{X} = (X^\psi, X^\theta, X^\phi) = (\psi, \theta, \phi)$ are Boozer coordinates with ψ the toroidal magnetic flux, θ and ϕ the poloidal and toroidal angles, respectively; ι is the rotational transform; $I = \int_0^\psi d\psi\,\iota(\psi)$; $B = B(\mathbf{X})$ is the magnetic field strength; $\mathbf{B} = (B_\psi(\mathbf{X}), B_\theta(\psi), B_\phi(\psi))$ are the 1-form (covariant vector) of the magnetic field; M, e, K and μ are the mass, charge, energy and magnetic moment of the particle; the longitudinal velocity $v_\parallel = \pm[2(K-\mu B)/M]^{1/2}$ is treated as a function of the independent variables ($\mathbf{X}$ and K); subscripts and superscripts denote covariant and contravariant components of vectors. We will not consider the evolution of the gyrophase in further treatment and so the magnetic moment will be a given quantity for a given particle.

We obtain the invariants perturbatively. With this aim, we decompose B into a certain part B_0 possessing some symmetry, which is considered as the "unperturbed" field, and the "perturbation" $\widetilde{B} = B - B_0$. Correspondingly, the differential form is split into an integrable part (the unperturbed system),

$$\boldsymbol{\gamma}_0 = \frac{e}{c}(X^\psi \mathrm{d}X^\theta - I\mathrm{d}X^\phi) + \frac{Mv_\parallel}{B_0}(B_\theta \mathrm{d}X^\theta + B_\phi \mathrm{d}X^\phi) - K\,dt, \tag{2}$$

and the part breaking the symmetry (the perturbation),

$$\boldsymbol{\gamma}_1 = -\frac{Mv_\parallel}{B_0^2}\,\widetilde{B}\,\frac{2K-\mu B_0}{2K-2\mu B_0}\,(B_\theta dX^\theta + B_\phi dX^\phi) + \frac{Mv_\parallel}{B_0} B_\psi dX^\psi. \tag{3}$$

Then we use the Lie transformation method [2, 3] to find a coordinate transformation which eliminates the asymmetry from the perturbed system to first order in the perturbation amplitude. The adiabatic invariants are constants of motion resulting from the approximate symmetry of the system.

The transformation in question is found by solving [3]

$$\boldsymbol{\gamma}_0 = \boldsymbol{\gamma}_0 + \boldsymbol{\gamma}_1 + \mathrm{d}\boldsymbol{\gamma}_0.\mathbf{H} - dS, \tag{4}$$

where S is an unknown scalar function, for the vector field $\mathbf{H}$ (generator of the transformation). The "corrected" coordinates are then found from

$$X_{1i} = X_i + \mathbf{H}(X_i). \tag{5}$$

3. Passing particles. We take the harmonic B_{00} of the Fourier expansion $B = \sum_{m,n} B_{mn} \exp(im\theta - in\phi)$ as the unperturbed part B_0 and all angle-dependent harmonics as the perturbation.

The unperturbed Lagrangian (2) can be written in the action–angle form,

$$\boldsymbol{\gamma}_0 = I_1(K,\psi)\,\mathrm{d}X^\theta + I_2(K,\psi)\,\mathrm{d}X^\phi - K\mathrm{d}t, \tag{6}$$

with the actions $I_1 \equiv \frac{e}{c}X^\psi + \frac{Mv_\parallel}{B_0}B_\theta$ and $I_2 \equiv -\frac{e}{c}I + \frac{Mv_\parallel}{B_0}B_\phi$, and the conjugate angles being θ and ϕ. Obviously, ψ is a constant of unperturbed motion in this case.

Substituting the perturbation (3) into Eq. (4) and solving for H_ψ, we obtain the "corrected" constant of motion corresponding to ψ:

$$\widetilde{\psi} = \psi + H_\psi = \psi + \frac{Mv_\parallel}{B_0^2}\,\frac{2K-\mu B_0}{2K-2\mu B_0}\sum_{m,n}\widetilde{B}_{mn}\,\frac{mB_\phi + nB_\theta}{m\iota - n}e^{im\theta - in\phi}. \tag{7}$$

The denominators in this equation ("resonance denominators") reflect the fact that the invariant does not work near resonant drift surfaces, where the topology of the phase space is destroyed by the perturbation.

4. Trapped particles. We assume that one of the magnetic field harmonics, (m_0, n_0), is larger than the rest and limits the particle motion along the field lines. Then the most essential features of the motion are described correctly by the unperturbed part if we take it in the form $B_0 = B_{00} - B_* \cos(m_0\theta - n_0\phi)$.

Let $\theta_1 = m_0\theta - n_0\phi$ and $\phi_1 = n_0\theta + m_0\phi$ be the new angular variables, which are more convenient. This ansatz yields $B_0 = B_0(\psi, \theta_1)$.

Equation (2) takes the form

$$\boldsymbol{\gamma}_0 = \left(\frac{e}{c}A_{\theta_1} + \frac{Mv_\parallel}{B_0}B_{\theta_1}\right)\mathrm{d}X^{\theta_1} + \left(\frac{e}{c}A_{\phi_1} + \frac{Mv_\parallel}{B_0}B_{\phi_1}\right)\mathrm{d}X^{\phi_1} - K\mathrm{d}t, \tag{8}$$

where A_α have been suitably defined. We have at once the action

$$I_2 = \frac{e}{c}A_\phi + \frac{Mv_\parallel}{B}B_\phi \tag{9}$$

(we drop from now on the subscripts on θ and ϕ). Let ψ_b be the turning point of the particle defined from $v_\parallel = 0$. Expanding I_2 in $\delta\psi = \psi - \psi_b$

$$I_2(\psi) \simeq \frac{e}{c}A_\phi + \frac{e}{c}A'_\phi\delta\psi + \frac{Mv_\parallel}{B_0}B_\phi, \tag{10}$$

we see that ψ_b is also a constant of motion. Transforming from ψ to ψ_b, neglecting the terms involving B'_0 and ι' (which implies that $\delta\psi \ll B/B', \iota/\iota'$) and the angular dependence of B_0 in $v_\parallel/B_0$, we find by integration over a period the action

$$I_1 \approx \frac{B_\theta A'_\phi - B_\phi A'_\theta}{2\pi}\,\frac{Mv_{\parallel max}}{B_0}\,\frac{8}{\kappa A'_\phi}\left[\mathbf{E} - (1-\kappa^2)\mathbf{K}\right], \tag{11}$$

where ${v_\parallel}^2 = {v_{\parallel max}}^2\kappa^{-2}(\kappa^2 - \sin^2\frac{\theta}{2})$, $\kappa^2 = (K - \mu B_{00} + \mu B_*)/(2\mu B_*)$ is the particle trapping parameter, $\mathbf{K} = \mathbf{K}(\kappa)$ and $\mathbf{E} = \mathbf{E}(\kappa)$ are the elipptic integrals of the 1st and 2nd kind. The conjugate angle variable ξ_1 is defined by

$$\kappa^{-1}\sin\frac{\theta}{2} = \mathrm{sn}\left(\frac{2\mathbf{K}}{\pi}\xi_1\right) = \mathrm{sn}\,\xi, \tag{12}$$

and the parallel velocity is $v_\parallel = v_{\parallel max}\,\mathrm{cn}\,\xi$. The angle conjugate to the action I_2 is $\xi_2 = \phi + \chi(\xi_1)$, where χ is a periodical correction determined from

$$\frac{\partial\chi}{\partial\xi_1} = \frac{2\kappa\mathbf{K}}{\pi}\,\frac{A'_\theta}{A'_\phi}\,\mathrm{cn}\,\xi. \tag{13}$$

Now we consider the perturbed field. Transforming Eq. (3) to to the action–angle variables, substituting it into (4) and collecting terms, we obtain

$$H_{\psi_b} = -\left(\frac{\partial I_1}{\partial\psi_b}\right)^{-1}\omega_1\sum_{k,l}\frac{kF_{\xi_2(kl)} - lF_{\xi_1(kl)}}{k\omega_2 - l\omega_1}e^{ik\xi_1 - il\xi_2}, \tag{14}$$

where F_i are defined by $\boldsymbol{\gamma}_1 = F_i\mathrm{d}X^i$, (k,l) are the numbers of Fourier harmonics in the canonical angles ξ_i, $\omega_1 = \left(\frac{\partial I_1}{\partial K}\right)^{-1} = \dot{\xi}_1$ and $\omega_2 = -\omega_1\frac{\partial I_1}{\partial\psi_b}\left(\frac{dI_2}{d\psi_b}\right)^{-1} = \dot{\xi}_2$ are the frequencies of the unperturbed motion (bounce and precession frequencies).

The necessary components of $\boldsymbol{\gamma}_1$ are

$$\begin{aligned} F_{\xi_1} &= -\frac{Mv_\parallel}{B_0^2}\tilde{B}\,\frac{2K - \mu B_0}{2K - 2\mu B_0}\left(B_\theta\frac{\partial\theta}{\partial\xi_1} + B_\phi\frac{\partial\phi}{\partial\xi_1}\right) \\ F_{\xi_2} &= -\frac{Mv_\parallel}{B_0^2}\tilde{B}\,\frac{2K - \mu B_0}{2K - 2\mu B_0}\left(B_\theta\frac{\partial\theta}{\partial\xi_2} + B_\phi\frac{\partial\phi}{\partial\xi_2}\right). \end{aligned} \tag{15}$$

The "corrected" constant of motion corresponding to ψ_b is

$$\tilde{\psi}_b = \psi + \frac{v_\parallel B_\phi}{\omega_{B0}A'_\phi} + H_{\psi_b}, \tag{16}$$

where $\omega_{B0} = eB_{00}/(Mc)$. I_1, which is the longitudinal invariant, receives a correction $\frac{\partial I_1}{\partial \psi_b} H_{\psi_b}$.

5. Conclusions. The invariants given by Eqs. (7) and (16), describe the particle motion if the particle is not too close to the region of transitioning particles or to the resonance drift surfaces (i.e., the drift surfaces where the denominators of these equations vanish). The terms with $n = 0$ in Eq. (7) and $l = 0$ in Eq. (16) represent the motion independent on the bounce phase, which can be described by the invariant [1] and the longitudinal adiabatic invariant, respectively. From Eq. (7) one can see that these terms usually dominate in devices with large $N = 5 - 10$. Indeed, as n is a multiple of number of periods, N, and ι is typically less than unity, the denominators of the $n \neq 0$ terms are always large in such devices. In low-N (compact) devices, the contribution of the terms with $n \neq 0$ may be significant.

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