On the theory of dust in a plasma using matched asymptotic expansions

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1. Introduction

The well-known ABR theory [1], originally developed for Langmuir probes, involves cold ions being accelerated radially towards a dust grain by a spherically symmetric potential. Using matched asymptotic expansions we exploit the smallness of $\epsilon_p$ (the size of the dust particle compared to $\lambda_D$) to find the electrostatic potential as a matched asymptotic expansion in $\epsilon_p$. This method gives new insights into the structure of the solution.

We consider a single dust grain suspended in an electropositive, collisionless plasma, with Maxwellian electrons. It is bombarded by ions and electrons from the plasma, and in equilibrium attains a negative charge.

2. The ABR Theory

Under an electrostatic potential $V(\rho)$, cold ions move radially towards the dust grain with velocity $v_i$ and density $n_i$. The governing equations are:

- Ion continuity: $\nabla \cdot \left( N_i v_i \right) = \frac{1}{\rho^2} \frac{\partial (\rho^2 N_i v_i)}{\partial \rho} = 0$
- Ion energy: $0 = \frac{1}{2} M v_i^2 + eV$
- Poisson: $\frac{d^2 V}{d \rho^2} + \frac{2}{\rho} \frac{d V}{d \rho} = -\frac{e}{\epsilon_0} \left[ N_i - N_0 \exp \left( \frac{eV}{kT_e} \right) \right]$

We use the normalisations:

$\phi = \frac{eV}{kT_e}$; $q = \frac{v_i}{v_B}$; $n = \frac{N_i}{N_0}$; $r = \frac{\rho}{\lambda_D}$;

where $v_B = \sqrt{\frac{2kT_e}{M}}$. The above equations then reduce to:

$\frac{d^2 \phi}{dr^2} + \frac{2}{r} \frac{d \phi}{dr} = e^\phi - \frac{J}{r^2 \sqrt{-\phi}}$

where $J$ is a constant, denoting the normalised ion current, given by:

$J = \frac{I_i}{n_0 e \sqrt{2kT_e/M 4\pi \lambda_D^2}}$
The boundary conditions are:

\[ \phi, \phi' \to 0 \text{ as } r \to \infty \]  \hspace{1cm} (3)

and on the dust surface, \( r = \epsilon_p \):

\[ \phi = \phi_p; \quad J = \alpha \epsilon_p^2 e^{\phi_p} \]  \hspace{1cm} (4)

where \( \alpha = \sqrt{\frac{M}{4 \pi m_e}} \). (4) represents the floating condition, i.e. balance of the ion and electron currents to the dust surface.

Points to note:

- We are interested in small dust particles, and the floating condition suggests this corresponds to small \( J \). Therefore we write \( J = \epsilon^2 \) for some \( \epsilon \ll 1 \).
- We will show \emph{a posteriori} that in fact \( \epsilon \sim \epsilon_p \sqrt{\alpha} \). This is consistent with values typically found in experiment.

The following theorem is fundamental:

\textbf{Uniqueness theorem [3]} Given the equation (2) and the boundary conditions (3), then for every \( J > 0 \) there exists a \emph{unique} solution \( \phi(r) \) for \( r \in (0, \infty] \).

Note that the floating boundary condition (4) does \emph{not} feature in this theorem. Instead, its role is to relate \( \epsilon_p \) to \( J \).

\section{Results}

The following structure emerges:

\begin{itemize}
  \item \( \phi^{(out)} \)
  \item \( \phi^{(in)} \)
  \item \( r \sim O(\epsilon^{\frac{2}{3}}) \)
\end{itemize}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{Spatial structure of the asymptotic ABR solution for small \( \epsilon \sim \epsilon_p \sqrt{\alpha} \).}
\end{figure}

where:

\[ \phi^{(out)} \sim -\frac{\epsilon^4}{2r^3} + \frac{\epsilon^8}{r^8} \left( -2 + \frac{24}{r^2} \right) + O(\epsilon^{12}) \]  \hspace{1cm} (5)

\[ \phi^{(in)} \sim \epsilon^{\frac{4}{3}} \left( \frac{c_0}{r_1} + c_1 + c_2 r_1^{\frac{1}{3}} \right) + \cdots \quad \text{as} \quad r_1 \to 0, \]  \hspace{1cm} (6)
with \( r = \epsilon \frac{2}{3} r_1 \), and:

\[
e_0 \simeq -1.3844; \quad c_1 \simeq 2.1699; \quad c_2 = \frac{4}{3} \left(-c_0\right)^{-\frac{1}{2}}
\]

Using the floating condition (4), it is seen that the dust surface lies in the \( r_1 \to 0 \) limit of the inner region. This confirms that indeed \( \epsilon \sim \epsilon_p \sqrt{\alpha} \), allowing us to transfer the parameterisation of the solution from \( \epsilon \) to \( \epsilon_p \).

For the typical case of Argon (\( \alpha \simeq 76.5 \)), fig. 2a shows the potential profile thus obtained for \( \epsilon_p = 0.01 \) whilst fig. 2b shows the variation of the floating potential with grain size:

\[\text{Figure 2: Plots of (a) potential profile and (b) dust floating potential vs. dust size for Argon.}\]

In physical terms, if \( r_p \) is the dust radius in unnormalised units we have a region of order \( r_p^{\frac{2}{3}} \lambda_D^\frac{1}{3} \alpha^\frac{1}{4} \) around the dust particle. This represents the shielding distance around the dust grain.

- Outside this region, as expected we have plasma-like behaviour, where \( n_i, n_e \gg \nabla^2 \phi \), i.e. quasineutrality.

- In the inner region, however, the space-charge becomes more important. The dust surface lies in the inner inner region, where \( \phi \) satisfies Laplace’s equation to lowest order.

4. Work in progress

We are now in a position to investigate the problem of a dust particle immersed in a flowing plasma. Let \( v_0 \) be the ion flow speed normalised by the Bohm velocity.
• For \( v_0 \ll 1 \), we can treat the solution as a perturbation of the solution found here. In fact, for \( v_0 < \epsilon^{\frac{2}{3}} \), we find that the inner region remains unchanged to lowest order in \( \epsilon \); only the outer solution loses spherical symmetry.

• For \( v_0 \sim \epsilon^{\frac{2}{3}} \) the problem decomposes into the same regions as before but the spherical symmetry of the inner region is broken by the stronger ion flow.

• For \( 1 > v_0 \gg \epsilon^{\frac{2}{3}} \) we expect a very different structure altogether.

This method will also be applied to the separate problem of a dust particle in the sheath, a typical scenario in terrestrial experiments. Here, the ion streaming velocity is at least \( v_B \) and the boundary conditions are changed.

5. References