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MHD Turbulence and Saturation of Small-Scale Dynamo

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I. Scaling Theories. The existing scaling theories of MHD turbulence are generalisations of Kolmogorov's 1941 dimensional theory of hydrodynamic turbulence. The latter can be summarised as follows. Assume scale invariance in the (inertial) range of scales $\ell_0 \gg \ell \gg \ell_\nu$ (ℓ_0 is the outer/forcing/system-size/energy-injection scale, ℓ_ν the inner/dissipation scale). Assume further that nonlinear interactions are mostly between motions ("eddies") of comparable scales (locality in k space). The energy distribution over scales is then deduced from the constancy of the energy flux in the inertial range:

$$\epsilon \sim u_\ell^2 \tau_\ell^{-1} = \text{const}, \tag{1}$$

where τ_ℓ is the energy transfer (cascade) time. Dimensionally, this time has to be the eddy-turnover time: $\tau_\ell \sim \tau_{\text{eddy}}(\ell) \sim \ell/u_\ell$. Then $\epsilon \sim u_\ell^3/\ell = \text{const}$, whence $u_\ell \sim \epsilon^{1/3}\ell^{1/3}$, or, for the isotropic spectrum, $E(k) \sim \epsilon^{2/3}k^{-5/3}$, the famous *Kolmogorov spectrum*. The scaling is fixed purely by dimensional analysis.

An analogous development for MHD turbulence proceeds as follows. Again assume scale invariance in the inertial range and locality in k space. Suppose that magnetic energy is dominated by a strong large-scale magnetic field B_0: this assumption will, in fact, not be justified unless there is an externally imposed mean field (see Sec. II), but it is worth exploring what it implies. Because of scale invariance, Eq. (1) remains valid, and the key question is what is the cascade time τ_ℓ. There are now two time scales corresponding to each "eddy" of size ℓ: the turnover time $\tau_{\text{eddy}}(\ell) \sim \ell/u_\ell$ and the Alfvén time $\tau_{\text{A}}(\ell) \sim \ell_\parallel/v_{\text{A}}$, where $v_{\text{A}} = B_0/\sqrt{4\pi\rho}$ is the Alfvén speed associated with the background field and $\ell_\parallel$ is the extent of the eddy along that field. The ratio of these time scales is a dimensionless combination, so, from dimensional analysis, $\epsilon \sim u_\ell^2\tau_{\text{eddy}}^{-1} f(\tau_{\text{A}}/\tau_{\text{eddy}}) = \text{const}$, where f is an arbitrary function. Thus, *dimensional analysis alone is not sufficient to determine the energy spectrum in MHD turbulence* and additional physical arguments are necessary.

The first such physical argument was devised by Iroshnikov and Kraichnan [1] (henceforth IK). Suppose MHD turbulence at small scales is a turbulence of interacting Alfvén-wave packets. Suppose also that it takes many interactions to destroy such a wave packet. A packet with amplitude u_ℓ takes time $\tau_{\text{A}}(\ell)$ to pass through a similar counter-propagating packet of parallel extent $\ell_\parallel$ (it is again assumed that only comparable scales interact). Over this time, its amplitude changes by $\delta u_\ell \sim \tau_{\text{A}} u_\ell^2/\ell$. The cascade time τ_ℓ is the time it takes for the cumulative change in amplitude to become comparable to u_ℓ. If the changes accumulate as a random walk, this implies $\delta u_\ell(\tau_\ell/\tau_{\text{A}})^{1/2} \sim u_\ell$, whence $\tau_\ell \sim \tau_{\text{eddy}}^2/\tau_{\text{A}}$. Substituting this into Eq. (1), we find

$$u_\ell \sim (\epsilon v_{\text{A}})^{1/4}\ell^{1/2}\ell_\parallel^{-1/4}. \tag{2}$$

A further assumption of isotropy at small scales, $\ell_\parallel \sim \ell$, gives the *IK spectrum* of MHD turbulence: $E(k) \sim (\epsilon v_{\text{A}})^{1/2}k^{-3/2}$. This had been the accepted view of MHD turbulence until direct numerical simulations became possible that showed that the isotropy

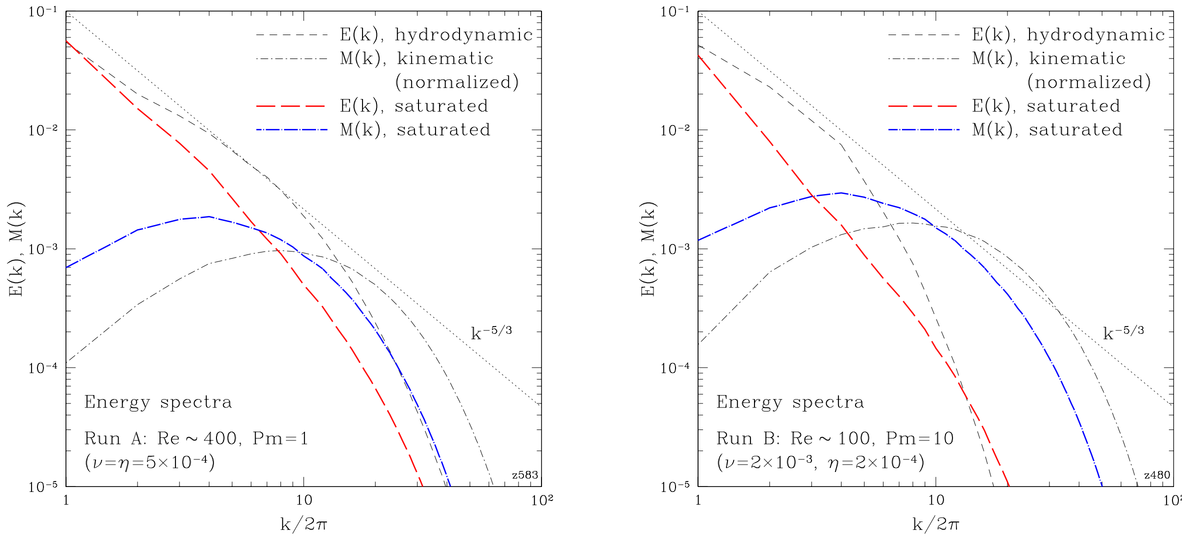

FIG. 1: Spectra of kinetic and magnetic energy for 256^3 simulations of isotropic MHD turbulence with $\mathrm{Pm} = 1$ (left panel) and $\mathrm{Pm} = 10$ (right panel). From Ref. [8].

assumption was definitely untrue and must be dropped (some of the most recent such simulations are [2, 3, 4]). The opposite extreme is to assume that there is no cascade in the parallel direction at all, i.e., $\ell_\| \sim \ell_0 \gg \ell$. Substituting this into Eq. (2) gives $E(k) \sim (\epsilon/\tau_\mathrm{A})^{1/2} k^{-2}$. This is the *spectrum of weak MHD turbulence* (e.g., [5, 6], henceforth WT).

In order for either of the above arguments to work, it is necessary that $\tau_\mathrm{A} \ll \tau_\mathrm{eddy}$, or, equivalently, $\tau_\mathrm{A} \ll \tau_\ell$: the Alfén-wave packets survive for many Alfvén times. In the case of the (isotropic) IK scaling, this condition is satisfied, $\tau_\mathrm{A}/\tau_\mathrm{eddy} \sim u_\ell/v_\mathrm{A} \ll 1$, and improves as ℓ decreases. On the contrary, for the (anisotropic) WT scaling, the ratio $\tau_\mathrm{A}/\tau_\mathrm{eddy}$ increases with decreasing ℓ and is order unity at the scale $\ell_* \sim \epsilon^{1/2}(\ell_0/v_\mathrm{A})^{3/2}$ (note that $\ell_* \ll \ell_0$ only if the background magnetic field is strongly above equipartition at the outer scale: $v_\mathrm{A} \gg u_{\ell_0}$). Below this scale, the argument breaks down. Goldreich and Sridhar [7] (henceforth GS) conjectured that, at all scales below ℓ_*, a strong-interaction regime is set up which satisfies the *critical balance* relation: $\tau_\mathrm{A}(\ell) \sim \tau_\mathrm{eddy}(\ell)$. Thus, the dimensionless ratio of these two time scales is 1 and Kolmogorov scaling is recovered, $E(k) \sim \epsilon^{2/3} k^{-5/3}$. With this scaling, the critical balance implies $\ell_\| \sim \ell_0^{1/3} \ell^{2/3}$, so there is a cascade in $\ell_\|$, but anisotropy persists at all scales.

II. Isotropic MHD Turbulence. Let us consider *forced nonhelical isotropic MHD turbulence*, i.e., suppose there is no externally imposed mean magnetic field and no net helicity in the forcing. Do the theories reviewed above give a qualitatively correct picture of such a turbulence? All of these theories make three key assumptions: (i) there is a strong background magnetic field with a spatial coherence length at or larger than the outer scale ℓ_0, (ii) at small scales, $\ell \ll \ell_0$, fluctuations are Alfvén waves propagating along this background field, so $B_\ell \sim u_\ell$ (scale-by-scale equipartition between magnetic and velocity fields), (iii) interactions are between comparable scales (locality in k space). None of these assumptions is, in fact, satisfied in the case of isotropic MHD turbulence. Let us examine this issue in more detail.

In the absence of a mean field, all magnetic fields are generated by the small-scale dynamo — a process of stretching and folding magnetic field lines by the random shear associated with the turbulent eddies. For a detailed account of the small-scale dynamo

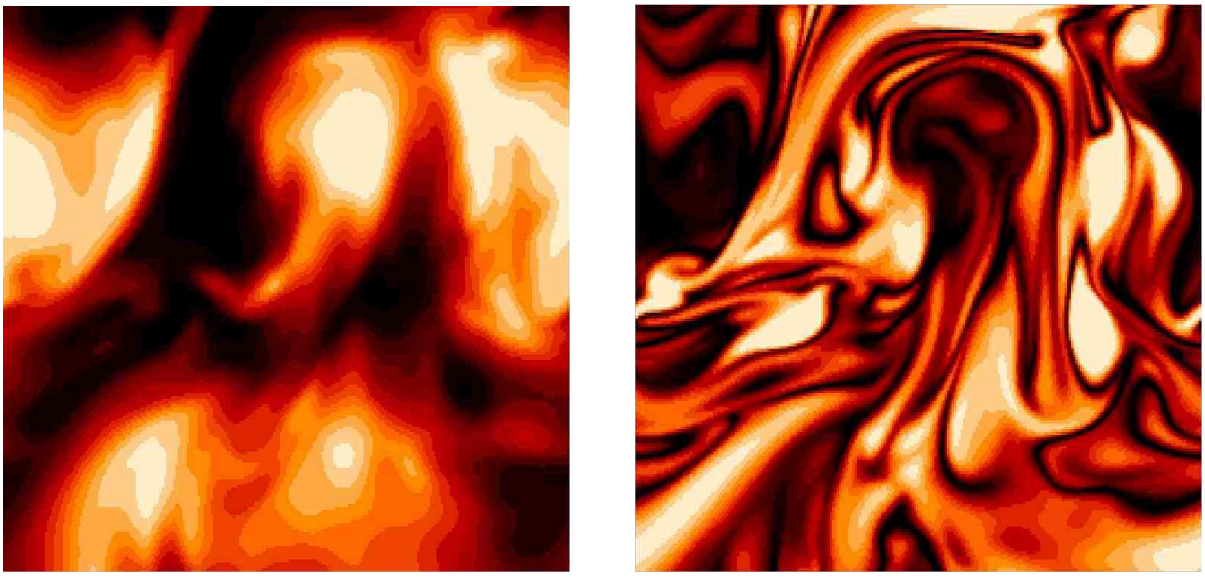

FIG. 2: Cross section of the absolute values of the velocity (left panel) and magnetic field (right panel) for the saturated state of Run B (see Fig. 1, left panel).

and all relevant references, we address the reader to Ref. [8]. Here we briefly reiterate some of the main points. In systems with magnetic Prandtl number $\mathrm{Pm} = \mathrm{Rm}/\mathrm{Re} \geq 1$, the small-scale dynamo is definitely present (given large enough Rm) and culminates in a nonlinear saturated MHD state with non-zero magnetic energy. The magnetic energy produced by the small-scale dynamo is concentrated at the resistive scale $\ell_\eta \sim \mathrm{Rm}^{-1/2}\ell_\nu$. The smallness of the field scale is due to spatial reversals of the field direction, while the curvature of the field lines is not large: the direction-alternating fields are organised in long thin *folds* whose length is comparable to the outer scale ℓ_0. The tendency to generate such a folded structure is simply a property of random advection in the kinematic regime [9, 10], but numerical simulations indicate that it survives in the nonlinear saturated state [8]. The folded structure is most pronounced in the limit $\mathrm{Pm} \gg 1$, when there is a clear disparity between the viscous and resistive scales ($\ell_\nu/\ell_\eta \sim \mathrm{Pm}^{1/2}$). The numerically more accessible case of $\mathrm{Pm} \gtrsim 1$, while non-asymptotic and, therefore, harder to interpret, retains the essential features of the large-Pm case. Fig. 1 shows kinetic and magnetic energy spectra for two such simulations. Neither these nor, indeed, larger [11] simulations of isotropic MHD turbulence show any evidence of the dominant magnetic energy being at the outer scale [assumption (i)] or of scale-by-scale equipartition $B_\ell \sim u_\ell$ [assumption (ii)]. The locality of interactions in k space [assumption (iii)] is harder to measure directly, but it is definitely violated by the very nature of small-scale dynamo: fields reversing direction at the resistive scale are generated by random shear of arbitrarily large scale. The fundamental difference in the structure of the velocity and magnetic fields is illustrated by Fig. 2.

Two questions naturally arise. First, what is the saturation mechanism of such a turbulence, i.e., what is the nature of the back reaction that the folded magnetic fields exert upon the turbulent flow? Second, what is the physics of small-scale fluctuations in the inertial range?

1. The Saturation Mechanism. The Lorentz tension force $\mathbf{B}\cdot\nabla\mathbf{B}$ responsible for the back reaction is quadratic in the magnetic field and only depends on its gradient along itself. Therefore, it does not "know" about the direction reversals within the folds. The folded structure provides the formally small-scale magnetic field with the degree of spatial coherence necessary for it to act back on the larger-scale turbulent velocities and thus resist further stretching. Using a statistical model of small-scale dynamo in a single-scale random flow, we have recently demonstrated [12] that *saturation can be achieved simply by partially suppressing the velocity gradients parallel to the local direction of the folds.* In so anisotropised a velocity field, the stretching of the field lines is weakened and a statistically stationary state is achieved as a balance between stretching, turbulent mixing, and Ohmic diffusion of the magnetic fields.

2. Alfvén Waves and Folded Fields. We propose that, in a multiscale turbulent flow, this balance is controlled by the velocity fluctuations at (or just below) the outer scale ℓ_0 [13, 8]. The length of the magnetic folds is ℓ_0, while the field reversal scale within the folds is $\ell_\eta \sim \mathrm{Rm}^{-1/2}\ell_0$. In the inertial range, $\ell_\nu \ll \ell \ll \ell_0$, the dynamics is controlled by Alfvén waves that propagate along the folds with the dispersion relation $\omega = \mathbf{kk} : \hat{\mathbf{b}}\hat{\mathbf{b}}\langle B^2\rangle$, where $\hat{\mathbf{b}} = \mathbf{B}/B$ [14]. Thus, we propose that *the saturated state of the isotropic MHD turbulence is a superposition of folded direction-alternating magnetic fields and Alfvén waves propagating along the folds.* The direction reversals account for the predominance of the small-scale modes in the Fourier spectra of the magnetic energy (Fig. 1). The Alfvén dynamics will determine the velocity spectra in the inertial range. If we replace the large-scale/mean magnetic field B_0 by $\langle B^2\rangle^{1/2}$ in the definition of the Alfvén speed, $v_\mathrm{A} = \langle B^2\rangle^{1/2}/\sqrt{4\pi\rho}$, it may be reasonable to expect that the scaling ideas reviewed in Sec. I should apply to the velocity fluctuations u_ℓ. Note that the turbulence cannot be weak in this case because v_A cannot be much larger than $u_{\ell_0} \sim \langle u^2\rangle^{1/2}$ and the condition $\tau_\mathrm{A} \ll \tau_\mathrm{eddy}$ is, thus, broken at all scales (see the end of Sec. I).

Numerical results that tentatively support the picture described above are presented in [8], but a definitive numerical study will require much higher resolutions.

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