Drift and acoustic modes in radially and axially inhomogeneous plasma

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Under laboratory conditions the equilibrium plasma can have a density gradient both along the magnetic field lines and in the perpendicular direction. In space plasmas, like in the solar coronal magnetic flux tubes, this axial density gradient may be due to stratification in the external gravity field, or due to any other externally imposed reason. The characteristic axial density scale length is usually much larger compared to the radial one, and in many situations the axial density profile may even be taken as constant. In those cases when the latter approximation is not physically justified, one encounters an eigenvalue problem in both radial and axial directions. Electrostatic waves propagating strictly parallel to the magnetic field lines will have all features of the ion acoustic (IA) wave as if the magnetic field were completely absent. However, waves propagating obliquely to the magnetic field lines, typically satisfying the condition $k_z/k_\perp \ll 1$, appropriate for a drift wave description, will combine both the drift and IA wave properties. The parallel IA component will then correspond to the classic case [1] dealing with the IA wave propagating along the density gradient.

In this work, we present exact analytic solutions for some equilibrium density profiles in such fully inhomogeneous plasmas. We consider a plasma embedded in the magnetic field $\vec{B}_0 = B_0\vec{e}_z$, with an equilibrium density gradient in both the directions perpendicular and parallel to the magnetic field. Density perturbations associated with an electrostatic wave propagating obliquely to the magnetic field are described by the ion continuity equation which can be written as

$$\frac{1}{n_0}\frac{\partial^2 n_1}{\partial t^2} + \nabla_\perp \cdot \frac{\partial \vec{v}_{i\perp 1}}{\partial t} + \frac{\partial}{\partial z}\frac{\partial v_{iz1}}{\partial t} + \frac{\partial \vec{v}_{i\perp 1}}{\partial t}\cdot\frac{\nabla_\perp n_0}{n_0} + \frac{\partial v_{iz1}}{\partial t}\frac{1}{n_0}\frac{\partial n_0}{\partial z} = 0. \quad (1)$$

The perpendicular and parallel ion velocities associated with the drift wave, in the usual limit $|\partial/\partial t| \ll \Omega_i$, are given by

$$\frac{\partial \vec{v}_{i\perp 1}}{\partial t} = \frac{1}{B_0}\vec{e}_z \times \nabla_\perp \frac{\partial \phi_1}{\partial t} - \frac{1}{\Omega_i B_0}\frac{\partial^2}{\partial t^2}\nabla_\perp \phi_1, \quad \frac{\partial v_{iz1}}{\partial t} = -\frac{e}{m_i}\frac{\partial \phi_1}{\partial z}. \quad (2)$$

We further use the quasi-neutrality condition, and omit the electron inertia effects so that electrons are described by the Boltzmann distribution. Assume a cylindric plasma that extends up to $r = r_0$ in the radial direction, with the equilibrium density $n_0 = n_0(r,z)$. Perturbations are of the form $f(r,z)\exp(-i\omega t + im\theta)$, where m is the poloidal drift-mode number, and we have

$\nabla_\perp = \vec{e}_r \partial/\partial r + \vec{e}_\theta \partial/(r\partial\theta)$. Eqs. (1)-(2) are combined yielding

$$\left[\frac{\partial^2}{\partial r^2} + \left(\frac{1}{r} + h_r\right)\frac{\partial}{\partial r} - \frac{m^2}{r^2} - \frac{r_0^2}{\rho_s^2} - \frac{\Omega_i}{\omega}\frac{m}{r}h_r - \frac{\Omega_i^2}{\omega^2}\left(\frac{\partial^2}{\partial z^2} + h_z\frac{\partial}{\partial z}\right)\right]\Phi(r,z) = 0. \quad (3)$$

Here, all spatial variables are in units of r_0, and $h_r = (\partial n_0/\partial r)/n_0$. As a good representative of various plasmas we consider a Gaussian equilibrium density profile in both directions $n_0(r,z) = N_0 \exp(-\kappa_r^2 r^2 - \kappa_z^2 z^2)$, where $\kappa_r \neq \kappa_z$ and typically $\kappa_r/\kappa_z \gg 1$, hence $h_r = -2\kappa_r^2 r$, $h_z = -2\kappa_z^2 z$. Introducing $\Phi(r,z) = \psi(r)\xi(z)$, from (3) we obtain the two equations

$$\left[\frac{\partial^2}{\partial r^2} + \left(\frac{1}{r} - 2\kappa_r^2 r\right)\frac{\partial}{\partial r} - \frac{m^2}{r^2} + b_0\right]\psi(r) = 0, \quad (4)$$

$$\left(\frac{\partial^2}{\partial z^2} - 2\kappa_z^2 z\frac{\partial}{\partial z} + a_0\right)\xi(z) = 0, \quad b_0 = \frac{2m\kappa_r^2\Omega_i}{\omega} - \alpha, \quad a_0 = \frac{r_0^2\omega^2}{c_s^2} - \frac{\alpha\omega^2}{\Omega_i^2}. \quad (5)$$

Here, α is an arbitrary constant which appears due to the separation of variables. The drift and IA modes are decoupled for $\alpha = 0$. In the homogeneous case, Eq. (5) yields the IA wave $\omega^2 = k_z^2 c_s^2$, otherwise it gives an axially dependent mode amplitude [1].

The general solutions of Eqs. (4) and (5) are, respectively,

$$\psi(r) = C_1 \cdot r^{-m} \cdot {}_1F_1\left[-\frac{b_0}{4\kappa_r^2} - \frac{m}{2}, 1-m, \kappa_r^2 r^2\right] + C_2 \cdot r^m \cdot {}_1F_1\left[-\frac{b_0}{4\kappa_r^2} + \frac{m}{2}, 1+m, \kappa_r^2 r^2\right], \quad (6)$$

$$\xi(z) = D_1 \cdot {}_1F_1\left[-\frac{a_0}{4\kappa_z^2}, \frac{1}{2}, \kappa_z^2 z^2\right] + D_2 \cdot z \cdot {}_1F_1\left[\frac{1}{2} - \frac{a_0}{4\kappa_z^2\Omega_i^2}, \frac{3}{2}, \kappa_z^2 z^2\right]. \quad (7)$$

In order to obtain physically well-behaved solutions, i.e., non-singular in the (r,z)-plain and even in the $\pm z$ directions, we here choose $C_1 = D_2 = 0$. Under laboratory conditions, the potential vanishes at least at $r = 0$, and $r = r_0$, resembling a radially standing drift wave. Due to the limited axial length in laboratory situations, a standing wave may appear in the axial direction too.

Eliminating α and find the dispersion equation in terms of the plasma parameters and a_0, b_0:

$$\omega^2\left(1 + b_0\frac{\rho_s^2}{r_0^2}\right) - \frac{2mc_s^2\kappa_r^2}{\Omega_i r_0^2}\omega - \frac{a_0 c_s^2}{r_0^2} = 0. \quad (8)$$

In physical units the dispersion equation takes the form $\omega^2(1 + b_0\rho_s^2) - 2m\Omega_i\kappa_r^2\rho_s^2\omega - a_0 c_s^2 = 0$. The parameter b_0 is to be determined from the requirement of vanishing solutions at r_0. Because Eq. (6) may contain oscillatory solutions, this in principle implies a sequence of discrete values for b_0 [2], [3]. As for a_0 it appears as a free parameter for an infinite plasma column, or it should be determined from the requirement $\xi(L/2) = 0$ for oscillatory solutions in the axial direction, where $\pm L/2$ determines the two ends of the plasma column. In the later case, we have only

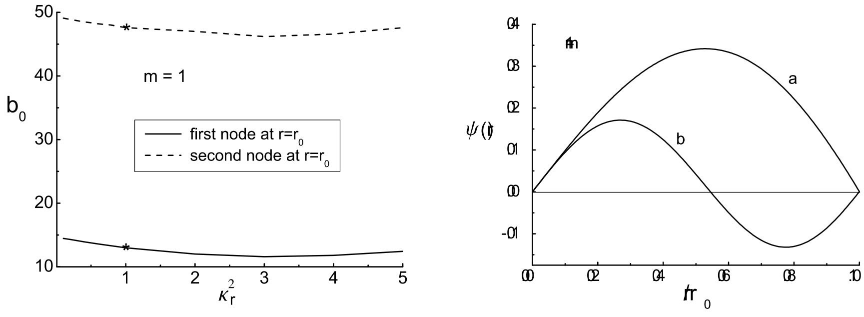

Figure 1: Left: Eigenvalues b_0 of Eqs. (6) (for the poloidal drift wave number $m = 1$) for various radial density profiles, yielding a standing wave solution in the radial direction. Right: The eigenfunctions Eqs. (6) (in arbitrary units) for $m = 1$, and for $\kappa_r^2 = 1$, $b_0 = 13$ (line a - the first eigenfunction), and $b_0 = 47.6$ (line b - the second eigenfunction). These parameters are denoted by $*$ in the eigen-values on Fig. 1.

a drift wave propagating in the poloidal direction, determined by the poloidal number m, and having a standing wave structure in both radial and axial directions.

As example, the lowest eigenvalues b_0 are determined for the poloidal wave number $m = 1$ (Fig. 1, left). Note that the density decreases with r so that for $\kappa_r^2 = 0.5, 1, 3$ we have $n_0(r_0)/N_0 = 0.61, 0.37, 0.05$, respectively. The corresponding eigen-functions, in arbitrary units, are presented in Fig. 1 (right).

In the axial direction, for a limited plasma column which extends e.g., up to $\pm 10 r_0$, the first term in the solution (7) for $\kappa_z^2 = 0.014$ gives the following first 4 eigenvalues: $a_0 = 0.013, 0.214, 0.609, 1.202$.

Note that for the above given κ_z the equilibrium density $n_0(z)/N_0$ decreases to $\simeq 0.24$ at $z/r_0 = 10$. These eigenvalues are used in the plot of the four even eigenfunctions in Fig. 2 (left). These show the normalized wave amplitude Φ and also the relative density perturbation $\hat{n}_1/n_0$, which (except for the lowest eigenvalue) increase along z in the presence of the given Gaussian equilibrium axial density profile. As an illustration, a standing wave pattern in the $r - z$ plane, associated with the drift mode traveling in the poloidal direction with the poloidal number m, is presented in Fig. 2 (right). Here, for $m = 1$. the lowest eigenfunction in the r-direction is combined with the 4th eigenfunction in the axial direction so that $\kappa_r = 1$, $b_0 = 13.3$, $\kappa_z^2 = 0.014$, $a_0 = 1.2015$, for $m = 1$.

To summarize, the behavior of coupled drift and ion acoustic modes is discussed in plasmas with density gradients perpendicular as well as parallel to the magnetic field lines. The perpendicular gradient of the density profile is a necessary ingredient for the existence of drift waves.

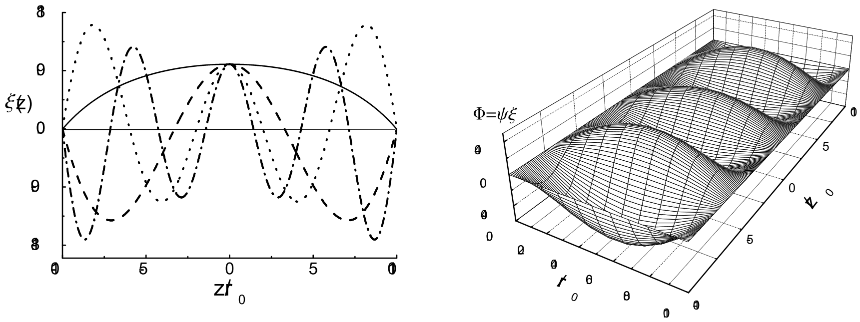

Figure 2: Left: The first four axial eigenfunctions Eq. (7) for $\kappa_z^2 = 0.014$, and for $a_0 = 0.013$ (full line), $a_0 = 0.214$ (dashed line), $a_0 = 0.609$ (dotted line), $a_0 = 1.202$ (dashed-dotted line). Right: The combination of the lowest eigenfunction in the r-direction with the 4th eigenfunction in the z-direction; $\kappa_r = 1$, $b_0 = 13.3$, $\kappa_z^2 = 0.014$, $a_0 = 1.2015$, and $m = 1$.

On the other hand, the density gradient in the direction of propagation of an IA wave is known to cause the growth of the IA wave potential amplitude and the relative density perturbation. The presence of both of these gradients in a nonlocal analysis implies solving both the perpendicular and axial eigenvalue problems. In the past, problems of that kind have been treated numerically, and separately for the axial and perpendicular directions. We have performed the wave analysis both in Cartesian and cylindric geometry and we have solved the appropriate eigenvalue differential equations *analytically* for a number of radial and axial density profiles. General solutions are found showing radially and axially varying wave amplitudes. The exact analytical solutions given here should be applicable in the description of the electrostatic drift-acoustic modes both in the laboratory and in space plasmas.

Acknowledgements: These results are obtained in the framework of the projects G.0304.07 (FWO-Vlaanderen), C 90205 (Prodex), GOA/2004/01 (K.U.Leuven), and the Interuniversity Attraction Poles Programme - Belgian State - Belgian Science Policy.

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