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The Vanishing of MHD Compressibility Stabilization in Closed Line Systems

A. J. Cerfon and J. P. Freidberg
MIT Plasma Science and Fusion Center, Cambridge MA 02139 USA

I. Introduction

It has been know since the early days of fusion research that plasma compressibility effects can stabilize ideal MHD interchange modes in closed line configurations. The present work focuses on this stabilizing mechanism using a more realistic plasma model than ideal MHD. One application of interest is LDX, which can be reasonably well modeled by a cylindrical hard-core Z-pinch. The hard core stabilizes $m \geq 1$ modes. Only the $m = 0$ sausage instability can be unstable. Ideal MHD predicts that for low β this mode can be stabilized by a sufficiently weak pressure gradient near the edge of the plasma. Specifically, stability follows if $rp'/p + 2\gamma > 0$ where γ = 5/3 is the ratio of specific heats representing the stabilizing effect of compressibility.

The present work reexamines the compressibility stabilization effect using a fluid model for electrons but with a full Vlasov treatment for the ions. There are two main results to report.
(1) First, an exact quadratic energy integral is derived that is valid for arbitrary 3-D static MHD equilibria, including both ergodic and closed field line configurations. This relationship shows that at marginal stability the compressibility stabilization term vanishes identically – there is no compressibility stabilization! This result is in contrast to other recent generalized theories [1, 2] which predict a modified form of compressibility stabilization but do not contain all the physics in the present model. (2) The second result is a derivation of the actual dispersion relation for a linear hard-core Z-pinch. The new model shows that instability persists for all negative values of rp'/p without any possibility of compressibility stabilization. It is demonstrated that the existence of resonant particles satisfying $\omega = k_\perp(V_{\mathbf{E}\times\mathbf{B}} + V_{\nabla B} + V_\kappa)$, is responsible for the persistence of instability, thereby explaining the absence or inclusion of compressibility stabilization in the Vlasov ion or ideal MHD models respectively.

II. The model

The model under consideration treats the electrons as a fluid and the ions with the Vlasov equation. The electrons are assumed to be a massless, collision dominated fluid with a scalar pressure and a simple energy equation. The electron model is given by

$$\begin{aligned} &\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{u}_e) = 0 && \text{Mass} \\ &en(\mathbf{E} + \mathbf{u}_e \times \mathbf{B}) + \nabla p_e = 0 && \text{Momentum} \\ &p_e = p_e(n) && \text{Energy} \end{aligned} \tag{1}$$

As stated the ions are described by the Vlasov equation.

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f + \frac{e}{m_i}(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \nabla_{\mathbf{v}} f = 0 \tag{2}$$

The model is closed with the low frequency Maxwell equations.

$$\begin{aligned} &\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \\ &\nabla \times \mathbf{B} = \mu_0 e \int (\mathbf{v} - \mathbf{u}_e) f d\mathbf{v} \\ &\nabla \cdot \mathbf{B} = 0 \end{aligned} \tag{3}$$

III. The general energy integral

We now consider the linear stability of MHD modes as described by the new model. Our main interest is to compare the stability predictions against those of ideal MHD for the usual case of static equilibria. In the Vlasov-fluid model static equilibria requires that the electrons carry all the current in equilibrium. This implies that the equilibrium ion distribution function satisfies $f = f(\varepsilon)$, where $\varepsilon = m_i \mathrm{v}^2/2 + e\phi(\mathbf{r})$. With these assumptions it can be shown that equilibrium force balance for the Vlasov-fluid model is given by $\mathbf{J} \times \mathbf{B} = \nabla p$ (where p is defined by $p = p_e + \int (m_i \mathrm{v}^2/3) f_i d\mathbf{v}$) which is identical to ideal MHD.

The next step is to linearize about this equilibrium. The connection to ideal MHD is made by introducing the electron fluid displacement vector: $\tilde{\mathbf{u}}_e = -i\omega\boldsymbol{\xi} + \mathbf{u}_e \cdot \nabla \boldsymbol{\xi} - \boldsymbol{\xi} \cdot \nabla \mathbf{u}_e$. After a lengthy calculation an exact energy integral can be derived, which has the form

$$|\omega|^2 = -\frac{\delta W_{VF}}{K_{VF}} \tag{4}$$

where

$$\delta W_{VF} = -\frac{1}{2}\int \boldsymbol{\xi}^* \cdot [\mathbf{J}\times\tilde{\mathbf{B}} + \tilde{\mathbf{J}}\times\mathbf{B} + \nabla(\boldsymbol{\xi}\cdot\nabla p)]d\mathbf{r}$$
$$K_{VF} = \frac{1}{2}\int \left\{ I_2 I_0 - |I_1|^2 + [\gamma_i/(\gamma_e+\gamma_i)]|I_1|^2 \right\}/I_0 \, d\mathbf{r} \qquad (5)$$
$$I_0 = -\int f_\varepsilon d\mathbf{v},\ I_1 = -\int \tilde{s} f_\varepsilon d\mathbf{v},\ \text{and}\ I_2 = -\int |\tilde{s}|^2 f_\varepsilon d\mathbf{v}$$

and the trajectory integral is $\tilde{s} = \int_{-\infty}^{t}[\boldsymbol{\xi}\cdot(\mathbf{E}+\mathbf{v}\times\mathbf{B}) - (\gamma_e p_e/en)\nabla\cdot\boldsymbol{\xi}]dt'$. Note that for $\partial f/\partial\varepsilon < 0$, $K_{VF} > 0$. The above relation, valid for both ergodic and closed line systems, shows that a plasma is marginally stable when $\delta W_{VF} = 0$, corresponding exactly to ideal MHD stability for incompressible displacements – there is no compressibility stabilization term in δW_{VF}. The presence of resonant particles strongly suggests that any change in plasma parameters will result in complex eigenvalues. Thus, the value of ω_i will be either positive (instability) or negative (stability) depending on the direction of change in the plasma parameters.

IV. The hardcore Z-pinch

We now explicitly demonstrate the destabilizing effects of resonant particles by calculating the dispersion relation for a hardcore Z-pinch, a cylindrical model of LDX. The equilibrium is characterized by $\mathbf{B} = B(r)\mathbf{e}_\theta$ and $p(r)$. We focus on the $m = 0$, $k_\perp \neq 0$ interchange mode which is stabilized by compressibility in ideal MHD. The analysis is greatly simplified by focusing on the parameter range corresponding to MHD modes: $\omega/\omega_{ci} \ll 1$, $k_\perp\rho_i \ll 1$, and$\beta \ll 1$. In this regime $B(r) \approx B(a)a/r$, and $\boldsymbol{\xi} = \boldsymbol{\xi}_\perp = \xi\mathbf{e}_r + \xi_z\mathbf{e}_z$. After another lengthy calculation a differential equation can be derived to determine the eigenfunction ξ and the eigenvalue ω.

$$\frac{\omega^2}{k_\perp^2}\frac{d}{dr}\left[\rho r^3 \frac{d}{dr}\left(\frac{\xi}{r}\right)\right] - \left[\omega^2\rho r^2 - 2r\frac{dp}{dr} - 4(\Gamma_e p_e + \Gamma_i p_i)\right]\xi = 0$$

$$\Gamma_e = \gamma_e\Omega(\bar{\Omega}Y^2 - 1)$$
$$\Gamma_i = \Omega(\bar{\Omega}^2 Y^2 - \bar{\Omega} - 1) \qquad (6)$$
$$\bar{\Omega} = \Omega + \Omega_* = r\omega_{ci}(\omega+\omega_*)/k_\perp \mathrm{v}_{Ti}^2$$
$$Y = -i\pi^{1/2}e^{-\bar{\Omega}}[1 + \Phi(i\bar{\Omega}^{1/2})]$$
$$\Phi(z) = (2/\pi^{1/2})\int_0^z exp(-\zeta^2)d\zeta$$

Using the familiar short wavelength local approximation, and assuming that $p_e = p_i = p/2$, two interesting limits can be calculated.

First, in the strongly MHD unstable regime (i.e. $-rp'/p \gg 1$ and $|\Omega| \gg 1$) the growth rate is given by

$$\frac{\omega_i^2}{\omega_M^2} \approx -2\left(\frac{rp'}{p} + \gamma_e + \frac{7}{4}\right) \tag{7}$$

where $\omega_M^2 = p/\rho r^2$ is the characteristic MHD frequency. We see that plasma compressibility enters in a manner very similar to ideal MHD but only in the strongly unstable regime where it is unimportant.

Second, for plasmas that would be ideal MHD stable one must take the opposite limit (i.e. $-rp'/p \ll 1$ and $|\Omega| \ll 1$) and it is here that the resonant particles play an important role. In this limit the plasma remains unstable, with a growth rate given by

$$\frac{\omega_i^2}{\omega_M^2} \approx 2\pi(k_\perp \rho_i)^2 \frac{\gamma_e^2(\gamma_e - 1)^3}{(\gamma_e + 1)^7}\left(-\frac{rp'}{p}\right)^5 \tag{8}$$

Curves of ω_i/ω_M vs. $-rp'/p$ for various $k_\perp \rho_i$ are illustrated in Fig. 1 using the local approximation. We see that there is no compressibility stabilization - the plasma is unstable for any $rp'/p < 0$ with the fastest growth rates occuring for short wavelengths.

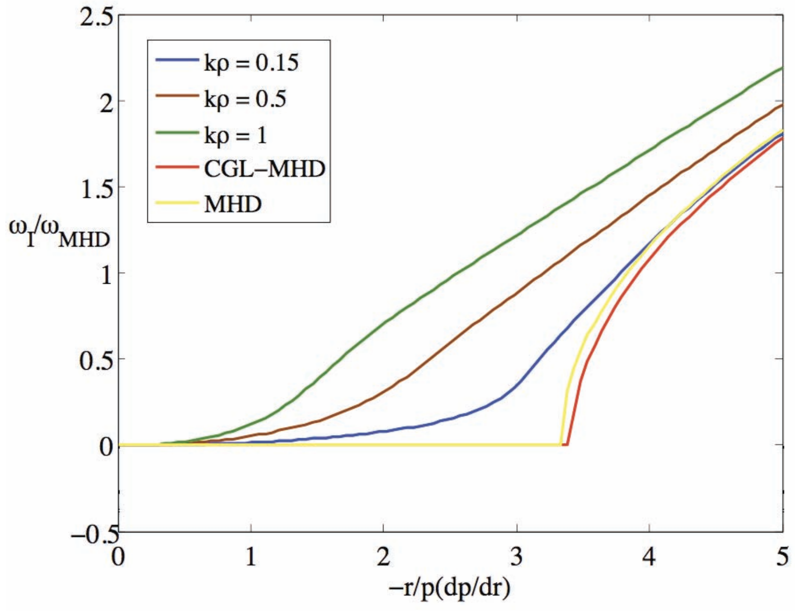

Figure 1: Growth rate vs. pressure gradient

References

[1] A. N. Simakov, P. J. Catto, and R. J. Hastie, Phys. Plasmas **8**, 4414 (2001)

[2] J. Kesner and R. J. Hastie, Phys. Plasmas **9**, 395 (2002)