35th EPS Conference on Plasma Phys. Hersonissos, 9 - 13 June 2008 ECA Vol.**32D**, P-2.180 (2008)

A sufficient condition for the linear stability of magnetohydrodynamic equilibria with field aligned incompressible flows

G.N. Throumoulopoulos[1](gthroum@uoi.gr), H. Tasso[2](het@ipp.mpg.de)

[1] *University of Ioannina, Association Euratom - Hellenic Republic, Division of Theoretical Physics, GR 45110 Ioannina, Greece*

[2] *Max-Planck-Institut für Plasmaphysik, Euratom Association, D-85748 Garching, Germany*

For static ideal magnetohydrodynamic (MHD) equilibria there is a powerful tool known as “the energy principle" providing necessary and sufficient conditions for linear stability [1]. In the presence of flow, however, the stability problem is much tougher because the force operator becomes non Hermitian; thus, only sufficient conditions were obtained [2]-[9]. Motivation of the present study is a couple of papers by Ilin and Vladimirov [6, 7] in which a sufficient condition was derived for the linear stability of plasmas with constant density and incompressible flows parallel to the magnetic field. This condition states that an equilibrium is stable to three dimensional perturbations provided that: i) the flow is sub-Alfvenic and ii) inequalities (51) of Ref. [7] are satisfied. Here we show, however, that those inequalities are not correct for the following reasons: i) The authors of Refs. [6] and [7] have not noticed that because of the field aligned flow the equilibrium current density lies on magnetic surfaces. This property simplifies the stability analysis and results in a single inequality for the sufficient condition in place of the couple of inequalities (51) of Ref. [7]. ii) A term associated with the flow shear was ignored in Refs [6] and [7]. The correct sufficient condition obtained here contains physically interpretable terms related to the magnetic shear and the flow shear.

We consider the steady states of a plasma of constant density and incompressible flow parallel to the magnetic field in the framework of ideal MHD (see for example Eqs. (1)-(6) of Ref. [10] written in convenient units and the density set to unity). Also, it is assumed the existence of well defined equilibrium magnetic surfaces in three dimensional geometry which are labelled by a smooth function ψ. Using

$$\mathbf{V} = \lambda \mathbf{B}, \tag{1}$$

where λ is an arbitrary function, the incompressibility condition ($\nabla \cdot \mathbf{V} = 0$) implies that λ is a surface quantity:

$$\lambda = \lambda(\psi). \tag{2}$$

Then, employing the identity $(\mathbf{V}\cdot\nabla)\,\mathbf{V} = \nabla V^2/2 - \mathbf{V}\times\nabla\times\mathbf{V}$, the momentum equation

$(\mathbf{V}\cdot\nabla)\mathbf{V} = \mathbf{J}\times\mathbf{B} - \nabla P$ leads to

$$\left(1-\lambda^2\right)\mathbf{J}\times\mathbf{B} = \nabla\left(P+\frac{\lambda^2 B^2}{2}\right) - B^2\nabla\left(\frac{\lambda^2}{2}\right), \tag{3}$$

where B is the magnetic field modulus. The component of (3) along the magnetic field implies that the quantity $P+\lambda^2 B^2/2$ is uniform on magnetic surfaces:

$$P+\frac{\lambda^2 B^2}{2} \equiv P_s(\psi). \tag{4}$$

Thus, owing to the flow the isobaric surfaces depart from the magnetic surfaces unlike the case of static equilibrium associated with the surface function $P_s(\psi)$. Consequently, Eq. (3) is put in the form $\left(1-\lambda^2\right)\mathbf{J}\times\mathbf{B} = P_s'\nabla\psi - (\lambda^2)'(B^2/2)\nabla\psi$ or

$$\mathbf{N} \equiv \mathbf{J}\times\mathbf{B} = g(\psi, B^2)\nabla\psi \tag{5}$$

where

$$g(\psi, B^2) \equiv \frac{P_s'}{1-\lambda^2} - \frac{(\lambda^2)'}{1-\lambda^2}\frac{B^2}{2}. \tag{6}$$

Eq. (5) implies that the current density lies on magnetic surfaces a property not noticed in Refs. [6] and [7]. Note that this holds because of the incompressible field aligned flows; for flows of arbitrary direction the current surfaces do not coincide with the magnetic surfaces. The fact that **B**, **J** and **V** share the same surfaces simplify the stability analysis to follow. To this end we also will need the quantity

$$\mathbf{M} \equiv \nabla\times\mathbf{N} = \nabla g\times\nabla\psi \tag{7}$$

from which it follows that

$$\mathbf{M}\cdot\mathbf{N} = 0. \tag{8}$$

In Refs. [6, 7] an energy principle was established for incompressible perturbations [$\nabla\cdot\xi(\mathbf{x},t)$=0] around a steady state and the following boundary conditions:

$$\mathbf{v}\cdot\mathbf{n} = \mathbf{b}\cdot\mathbf{n} = \xi\cdot\mathbf{n} = 0. \tag{9}$$

Here $\mathbf{v}(\mathbf{x},t)$ and $\mathbf{b}(\mathbf{x},t)$ are the perturbations of the velocity and the magnetic field and conditions (9) are imposed on a fixed boundary $\partial\mathscr{D}$ surrounding the plasma domain $\mathscr{D}$. Therefore, internal modes are considered. The principle is based on the fact that the perturbation energy

$$E \equiv \int_{\mathscr{D}}\left(\frac{1}{2}\dot{\xi}^2 - \frac{1}{2}\xi\cdot\hat{K}\xi\right)dV, \tag{10}$$

is conserved by the linearized ideal MHD equations ($dE/dt = 0$). Here $\hat{K}$ is a symmetric operator defined by the formula $\hat{K}\xi = \mathbf{V}\times\nabla\times\mathbf{v} + \mathbf{v}\times\Omega - \mathbf{B}\times\nabla\times\mathbf{b} - \mathbf{b}\times\mathbf{J}$, where $\mathbf{v} = \nabla\times(\xi\times\mathbf{V})$,

$$\mathbf{b} = \nabla\times(\xi\times\mathbf{B}), \tag{11}$$

and $\Omega = \nabla\times\mathbf{V}$. Evidently, E as a quadratic functional of $\dot{\xi}$ and ξ is positive definite if the potential energy

$$W = -\frac{1}{2}\int_{\mathscr{D}} \xi\cdot\hat{K}\xi dV \tag{12}$$

is positive definite. It is known, however, that for flows of arbitrary direction the functional W is never strictly positive definite [2]-[9]. For this reason further consideration is restricted to the steady states with field aligned flows described by (1. In this case (12) can be written in the form

$$W = \frac{1}{2}\int_{\mathscr{D}} \left\{(1-\lambda^2)\left[\mathbf{b}^2+\mathbf{b}\cdot(\mathbf{J}\times\xi)\right]-2\lambda(\xi\cdot\nabla\lambda)\left[\xi\cdot(\mathbf{B}\cdot\nabla)\mathbf{B}\right]\right\}dV. \tag{13}$$

Derivation of (13) is given in Ref. [6]. Whenever the potential energy (13) is positive definite the equilibrium is linearly stable.

As in Refs. [6] and [7] assuming that $\mathbf{J}\times\mathbf{B}\neq 0$ we express the perturbation vector ξ in the form

$$\xi = \alpha(\mathbf{x},t)\mathbf{N}+\beta(\mathbf{x},t)\mathbf{J}+\gamma(\mathbf{x},t)\mathbf{B}. \tag{14}$$

It can then be shown (see Appendix of Ref. [11]) that W assumes the form

$$W = W_1 + W_2, \tag{15}$$

$$W_1 = \frac{1}{2}\int_{\mathscr{D}}(1-\lambda^2)(\mathbf{b}+\alpha\mathbf{J}\times\mathbf{N})^2 dV, \quad W_2 = \int_{\mathscr{D}} A\alpha^2, \tag{16}$$

where

$$A = -(1-\lambda^2)(\mathbf{J}\times\mathbf{N})\cdot(\mathbf{B}\cdot\nabla)\mathbf{N}-\lambda(\mathbf{N}\cdot\nabla\lambda)\left(\mathbf{N}\cdot\frac{\nabla B^2}{2}+N^2\right). \tag{17}$$

Evidently, W is positive semidefinite if $|\lambda|\leq 1$ and

$$A \geq 0 \ \text{ in } \ \mathscr{D}. \tag{18}$$

Inequality (18) is substantially different from the respective inequalities (51) of Ref. [7]. In particular, the last term of (17) containing $\nabla\lambda$ was missed in [6] and [7]. Using the equilibrium relations (2) for λ and (5) for $\mathbf{N}$, (17) reduces to

$$\begin{aligned} A &= -g^2\Big\{(1-\lambda^2)(\mathbf{J}\times\nabla\psi)\cdot(\mathbf{B}\cdot\nabla)\nabla\psi \\ &\quad +\frac{(\lambda^2)'}{2}|\nabla\psi|^2\left(\nabla\psi\cdot\frac{\nabla B^2}{2}+g|\nabla\psi|^2\right)\Big\}. \end{aligned} \tag{19}$$

Taking into account Eqs. (15)-(16) and (19) we can conclude that *a general steady state of a plasma of constant density and incompressible flows parallel to the magnetic field is stable to small three-dimensional perturbations if i) the flow is sub-Alfvénic and ii)*

$$\tilde{A} \equiv A/g^2 \geq 0. \tag{20}$$

Using the relation $(\mathbf{B}\cdot\nabla)\nabla\psi = \mathbf{J}\times\nabla\psi - (\nabla\psi\cdot\nabla)\mathbf{B}$, $\tilde{A}$ can be put in the physically interpretable form:

$$\begin{aligned} \tilde{A} &= -(1-\lambda^2)\left[(\mathbf{J}\times\nabla\psi)^2 - (\mathbf{J}\times\nabla\psi)\cdot(\nabla\psi\cdot\nabla)\mathbf{B}\right] \\ &\quad -\frac{(\lambda^2)'}{2}|\nabla\psi|^2\left(\nabla\psi\cdot\frac{\nabla B^2}{2} + g|\nabla\psi|^2\right). \end{aligned} \tag{21}$$

Since $\lambda^2 < 1$ the first term in (21) containing $(\mathbf{J}\times\nabla\psi)^2$ is negative and therefore destabilizing. We conjecture that this is related to current driven modes. The other terms can be either stabilizing or destabilizing. Specifically, the second term containing $(\nabla\psi\cdot\nabla)\mathbf{B}$ depends on the differential variation of **B** perpendicular to the magnetic surfaces and therefore it relates to the magnetic shear. The last two terms depending on $(\lambda^2)'$ are connected with the velocity shear [Eq. (1)]. The third term relates to the magnetic shear through the differential variation of B^2 perpendicular to the magnetic surfaces in association with the term $\nabla\psi\cdot\nabla B^2$. At last, the fourth term has an additional implicit dependence on $(\lambda^2)'$ and P_s' through the quantity g [Eq. (6)].

It is recalled that the sufficient condition established here can be applied to any steady state without geometrical restriction. Application to steady states of fusion concern in connection with possible stabilizing effects of the flow is under way.

References

[1] I. B. Bernstein, E. A. Frieman, M. D. Kruskal, R. M. Kulsrud, Proc. Roy. Soc. (London), **A244**, 17 (1958).

[2] E. Frieman and M. Rotenberg, Rev. Mod. Phys. **32**, 898 (1960).

[3] S. Friedlander, M. M. Vishik, Chaos **5**, 416 (1995).

[4] H. Tasso, Phys. Lett. A **222**, 97 (1996).

[5] E. Hameiri, Phys. Plasmas **5**, 3270 (1998).

[6] V. A. Vladimirov and K. I. Ilin, Phys. Plasmas **5**, 4199 (1998).

[7] K. I. Ilin and V. A. Vladimirov, Phys. Plasmas **11**, 3586 (2004).

[8] V. I. Ilgisonis and I. V. Khalzov JETP Letters **82**, 570 (2005).

[9] M. Hirota, Z. Yoshida, E. Hameiri Phys. Plasmas **13**, 022107 (2006).

[10] H. Tasso and G. N. Throumoulopoulos, Phys. Plasmas **8**, 2378 (1998).

[11] G. N. Throumoulopoulos, H. Tasso, Phys. Plasmas **14**, 122104 (2007).