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Nonlinear dust-lattice waves: a modified Toda lattice

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Abstract

Large-amplitude oscillations and waves in dust-lattices are investigated, employing techniques used in Toda lattice analysis. The lattice consists of a periodic linear chain of particles, or a ring as occurs in experimentally observed dust particle clusters. Longitudinal waves in a 3-particle cluster are investigated, with a numerical solution of the nonlinear equations of motion. For large total energy in the system, the particle motion is found to have a triangular waveform. Regular motion is found for small amplitudes, but chaotic motion appears to be present for large amplitudes.

Introduction

Charged dust grains in a plasma interact with a Coulomb potential, but also with an exponential component to the potential, due to Debye shielding in the background plasma. The modes of vibration of charged-dust lattices in a plasma have been extensively studied in the linear limit. Both longitudinal and transverse modes have been identified (eg [1]). The oscillatory modes of clusters of small numbers of dust particles have also been studied (eg [2]). Some work on small-amplitude nonlinear dust-lattice waves has also been done (eg [3, 4]). Lattice particles that interact purely via an exponentially decaying potential constitute the “Toda lattice" [5, 6, 7]. The Toda lattice is of considerable general interest in nonlinear dynamics because it is an example of a completely integrable Hamiltonian system, with analytic soliton solutions [8, 9]. This is in contrast to many other systems which show regions of stochasticity in their solutions. Prior to the proof of integrability, only regular numerical solutions of the Toda lattice were found for all energies [10], for a 3-particle periodic lattice with nearest neighbour interactions. An additional isolating integral (constant of the motion) for the Toda lattice was found by Henon [8].

Here we analyse the nonlinear motions of a 3-dust-particle periodic lattice, interacting via both the Coulomb and exponentially decaying potential, employing the same techniques as have been used for the Toda lattice. Phase plots show whether the motion falls on smooth invariant curves, the trajectories are regular, and the motion is integrable, or whether there is non-regular, nonintegrable stochastic motion. The results are of relevance to oscillations of both linear dust-lattices and ring-form dust clusters.

The 3-particle System

Consider 3 dust particles, with the same mass and (negative) charge, constrained to move on a ring, with equilibrium separation of r_0. At a point in their motion, their coordinates become r_1, r_2 and r_3, as shown in Figure 1(a). This system can also represent the longitudinal motions of a linear periodic lattice of 3 particles. Nearest neighbour interactions between adjacent particles on the ring only are included.

The interaction potential includes Coulomb repulsion as well as plasma Debye shielding. The Hamiltonian may then be written in normalized form:

$$H = \frac{1}{2}\left(p_1^2 + p_2^2 + p_3^2\right) + \frac{\exp[-(r_1 - r_3)/\lambda_D]}{(r_1 - r_3)} + \frac{\exp[-(r_2 - r_1)/\lambda_D]}{(r_2 - r_1)} + \frac{\exp[-(r_3 - r_2)/\lambda_D]}{(r_3 - r_2)} - 3\exp[-1/\lambda_D]. \quad (1)$$

Here the scale length is r_0, and the Debye length (scaled to r_0) is λ_D. The final term sets H to zero at the equilibrium. The Toda Hamiltonian is obtained by retaining just the exponential dependence of the interaction potentials.

There is an obvious isolating integral for the system, namely, the total momentum. We can therefore transform to the rotating system in which the total momentum is zero. As was employed for the Toda lattice [9, 10], a linear transformation of the particle coordinates may then be made to eliminate one space coordinate and preserve the form of the kinetic energy, exploiting the periodicity of the system,

$$r_1 = 2x/3^{1/2} + 2y, \quad r_2 = 1 - 4x/3^{1/2}, \quad r_3 = 2 + 2x/3^{1/2} - 2y, \quad (2)$$

yielding the Hamiltonian for the remaining 2 degrees of freedom:

$$\bar{H} = \frac{1}{2}\left(p_x^2 + p_y^2\right) + \frac{\exp[-1/\lambda_D]}{24}\left[\frac{\exp[(2y + 2\sqrt{3}x)/\lambda_D]}{(1 - 2y - 2\sqrt{3}x)} + \frac{\exp[(2y - 2\sqrt{3}x)/\lambda_D]}{(1 - 2y + 2\sqrt{3}x)} + \frac{\exp[-4y/\lambda_D]}{(1 + 4y)}\right] - \exp[-1/\lambda_D]/8. \quad (3)$$

A solution for the 2 space and 2 momentum coordinates is obtained numerically by integrating the equations of motion,

$$\dot{p}_x = -\frac{\partial \bar{H}}{\partial x}, \quad \dot{p}_y = -\frac{\partial \bar{H}}{\partial y}, \quad \dot{x} = \frac{\partial \bar{H}}{\partial p_x}, \quad \dot{y} = \frac{\partial \bar{H}}{\partial p_y}, \quad (4)$$

from an initial (y, p_y) point, and plotting the intersections of a trajectory with a Poincaré surface of section, which we take to be $x = 0$, the (y, p_y) plane. If these intersections always fall on

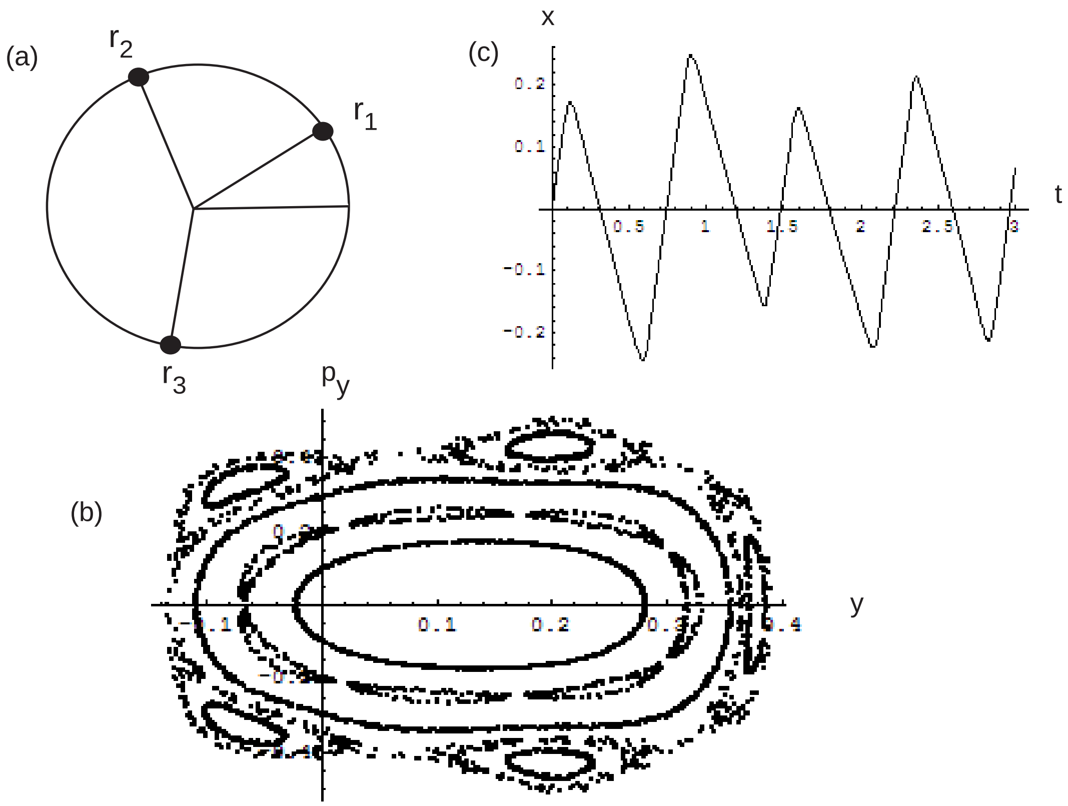

Figure 1: (a) The ring of 3 particles (b) A typical waveform (c) The (y, p_y) Poincaré sections for the motions corresponding to energy $E = 1$.

smooth invariant curves, the trajectories are regular, and the motion is integrable. Numerically found intersections of the trajectories with the $y - p_y$ plane for a given energy ($\bar{H} = E = 1$), and $\lambda_D = r_0$, are shown in Figure 1(b). Each family of intersections corresponds to different values of the initial conditions. A typical triangular waveform is shown in Figure 1(c). Regular trajectories are found for small amplitude oscillations of the grains, but there are indications that chaotic trajectories occur for large amplitude oscillations, in contrast to the Toda lattice. An isolating integral has not been found.

Conclusion

It appears that the nonlinear motion of this system is regular for small amplitude, but chaotic for large amplitude oscillations. An isolating integral has not been found. Experimental verification of these results, including the triangular waveforms, may be sought in experiments on linear dust lattices or in few-particle clusters, although in the experiments there are many complicating factors such as the influence of ion flow past the grains. We note also that purely transverse oscillations of the particles involve, in the first approximation, an inverse exponential dependence of the potentials on the square of the distance between particles, as well as a background confining potential, in contrast to the linear dependence of the Toda lattice and the system with purely longitudinal motions, while coupled longitudinal and transverse oscillations involve a combination of the 2 types of potential.

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