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An MHD compatible model for self-organized criticality in toroidally confined plasma

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Abstract

We present a self-organized criticality (SOC) model for the magnetic field in toroidal confinement devices. The model is based on a Cellular Automaton (CA), and it is fully compatible with MHD. The physics it implements is a magnetic field driven by a toroidal current, with resistive instabilities that are relaxed in local diffusive events. Results are presented concerning the magnetic field structure in the poloidal plane, magnetic flux surfaces, safety-factor profiles, and the dissipation of magnetic energy during SOC state.

Introduction

Several Self-Organized Criticality (SOC) models for confined plasmas have been suggested, which with success model some aspects of plasma turbulence (e.g. [1, 2, 3, 4]). These models all use the sand-pile analogy, with variables such as the local height of the sand-pile, height differences across the pile etc.

Our aim is to construct a SOC model for the usual physical variables (e.g. magnetic field), which is physically interpretable in a consistent way, and where we do not make use of the sand-pile analogy at all. Here, we introduce a SOC model for the magnetic field in the form of a cellular automaton (CA) that is compatible with MHD, and we present an application of the model to a toroidal confinement device (reversed field pinch or tokamak).

The model

To achieve MHD compatibility in the CA model, we use the vector potential $\mathbf{A}$ as the grid variable, and in order to calculate derivatives, $\mathbf{A}$ is interpolated, which allows to determine $\mathbf{B} = \nabla \times \mathbf{A}$ and $\mathbf{J} = (\mathbf{c}/\mathbf{4}\pi)\nabla \times \mathbf{B}$ in the usual MHD way, so that e.g. $\nabla \cdot \mathbf{B} = \mathbf{0}$ is ensured.

In the application to the magnetic field of a toroidal confinement device, we concentrate on the 2-D poloidal plane and use a 2D grid. As initial condition $\mathbf{A^0}$, we use a simple, analytical, equilibrium solution (in cylindrical coordinates, R, θ, Z), with $A^0_R = 0$ and

$$
\begin{aligned}
A^0_\varphi(R,Z) &= \frac{B_0}{R} \int \frac{d\xi}{\bar{q}_s(r(\xi))} \\
A^0_Z(R) &= -B_0 R_0 \ln R
\end{aligned}
$$

(with $\xi = r^2/2$, $r^2 = (R-R_0)^2 + Z^2$, and with q_s the safety factor), which yields simple circular flux surfaces.

The current of the initial field can be written as $\mathbf{J} \propto \nabla \times \mathbf{B} = \nabla \times \nabla \times \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$, and for the chosen $\mathbf{A^0}$, $\nabla \cdot \mathbf{A^0} = \mathbf{0}$ holds, so that $\mathbf{J} \propto -\nabla^2 \mathbf{A^0}$. In cylindrical coordinates then, we have $J_R^0 = 0$, $J_Z^0 = 0$, and

$$J_\varphi^0 \propto -\partial_Z^2 A_\varphi^0 - \partial_R \left[\frac{1}{R} \partial_R \left(R A_\varphi^0 \right) \right]. \tag{1}$$

It thus follows that A_Z yields the toroidal field B_φ, physically due to the external coils, and A_φ yields the poloidal field B_p, physically created by the toroidal current J_φ. The driving scenario of the system is thus that the toroidal magnetic field from the external coils is not changed, implying that A_z stays constant, the toroidal current though is driven in order to build up the poloidal field, which implies that A_φ evolves such that J_φ increases.

Translating this scenario to CA rules, we let A_Z = constant, and we systematically increase A_φ by adding increments to it,

$$A_\varphi(t+1, \vec{x}_{ij}) = A_\varphi(t, \vec{x}_{ij}) + \delta A_\varphi(t, \vec{x}_{ij}),$$

at one (usually random) grid site i, j, at a time, and where the increments are in the direction of the unperturbed field, $\delta A_\varphi = s A_\varphi^{(0)}(R, Z)$, with s a constant or a random number. In this way, the toroidal current increases because either A_φ increases or its local curvature increases.

Regarding the instability that is needed in a SOC model, we implement a resistive, current driven instability, making use of the MHD induction equation in terms of the vector potential, $\partial \mathbf{A}/\partial \mathbf{t} = \mathbf{v} \times \mathbf{B} + \eta(\mathbf{c^2}/4\pi)\nabla^2 \mathbf{A} - \eta(\mathbf{c^2}/4\pi)\nabla(\nabla \cdot \mathbf{A}) + \nabla \chi$, with η the resistivity and χ an arbitrary function. For the $\mathbf{A^0}$ chosen, $\nabla \cdot \mathbf{A^0} = \mathbf{0}$ and $\mathbf{J} \propto -\nabla^2 \mathbf{A}$, as mentioned above, and if we neglect the convective term and χ, we find that $\mathbf{A}$ evolves as $\partial \mathbf{A}/\partial \mathbf{t} = \eta(\mathbf{c^2}/4\pi)\nabla^2 \mathbf{A} \propto -\eta(\mathbf{c^2} 4\pi)\mathbf{J}$, which for A_φ alone writes as

$$\partial A_\varphi/\partial t = -\eta(c^2/4\pi)(\nabla^2 \mathbf{A})_\varphi \propto \eta(\mathbf{c^2}/4\pi)\mathbf{J}_\varphi. \tag{2}$$

Obviously, the current causes a diffusive evolution of A_φ if η is finite. We thus implement the physical scenario of threshold dependent local diffusion. First, we define a threshold J_{cr}. If the current is below the threshold, $|J_\varphi| < J_{cr}$, nothing happens ($\eta = 0$ in Eq. (2)). If the current exceeds the threshold, $|J_\varphi| > J_{cr}$, local diffusion sets in, according to Eq. (2), i.e. η has become finite (anomalous resistivity).

Translating this to CA rules, we first turn to Cartesian coordinates ($A_\varphi \to A_y$, $J_\varphi \to Jy$), where $J_y \propto -\nabla^2 A_y$, and we use a difference scheme approximation for $\nabla^2 A_y$, $\nabla^2 A_{y;i,j} \approx (A_{y;i+1,j} -$

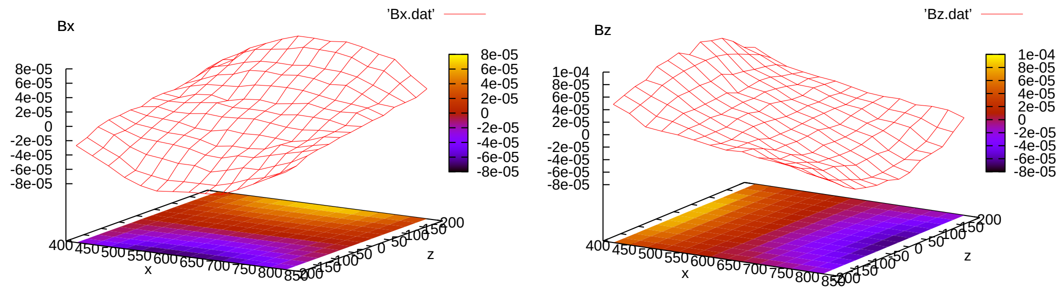

Figure 1: The poloidal magnetic fields B_x and B_z in the poloidal cross-section during SOC state.

$2A_{y;i,j} + A_{y;i-1,j}) + (A_{y;i,j+1} - 2A_{y;i,j} + A_{y;i,j-1})$, in which we change the factors and sign and define

$$dA_{y,ij} := A_{y,ij} - \frac{1}{4}\sum_{n.n.} A_{y,n.n.}$$

as the substitute for and approximation to the current. The sum is over the four nearest neighbours (n.n.) in the 2D rectangular grid, so that $dA_{y,ij}$ is the difference between the central value and the mean of its 4 neighbours. We now consider an instability to occur if $|dA_{y,ij}| > A_{cr}$ (as a substitute for $|J_\varphi| > J_{cr}$).

Physically, the local instability is relaxed since A_φ locally diffuses according to the MHD induction equation, Eq. (2), whereby the diffusion process removes the cause of the diffusion, the current evolves from super-critical ($|J_y| > J_{cr}$) to sub-critical ($|J_y| < J_{cr}$). The corresponding CA rule should mimic this in terms of $dA_{y,ij}$, the current substitute in the CA, such that instabilities with $|d_{Ay,ij}| > A_{cr}$ are relaxed to a state $|dA_{y,ij}| < A_{cr}$. We apply the redistribution rules $A_{y,ij}(t+1) \longrightarrow A_{y,ij}(t) - \frac{4}{5} dA_{y,ij}$ for the central point, and $A_{y,n.n.}(t+1) \longrightarrow A_{y,n.n.}(t) + \frac{1}{5} dA_{y,ij}$ for the nearest neighbours, which imply that $dA_{y,ij}(t+1) = 0$, i.e. the instability indeed is relaxed.

The free parameters of the model are the threshold A_{cr}, and the safety factor profile q_s, for which in the following we throughout use $q_s \equiv 2$.

Results

Starting from the initial conditions, the system goes through a transient phase and then reaches the SOC state as a dynamical equilibrium. In SOC state, the toroidal field B_φ falls off as $1/R$. The poloidal fields, shown in Fig. 1, exhibit a characteristic shape, including a change of sign. Actually, B_x and B_y stay very close to these characteristic shapes in SOC state, they just slightly fluctuate about them. The stiffness of the magnetic field structures is very high, performing a pure off-axis driving experiment, we find that, within small fluctuations, the same

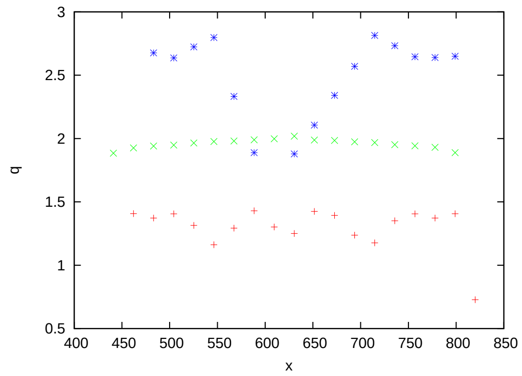

Figure 2: The safety factor profile along the minor radius for the initial state (×, green), and in the SOC state for the standard threshold value (+, red), and for half the standard threshold (*, blue).

poloidal magnetic topology is reached as in the uniform driving case.

A typical safety factor profile q_s during SOC state is shown in Fig. 2, as calculated numerically from the definition $q_s = \frac{1}{2\pi}\int_C \frac{1}{R}\frac{B_\varphi}{B_p}\,ds$. It is found that the safety factor profile depends on the threshold, because B_p is proportional to threshold and B_φ is unchanged from its initial values.

Finally, determining the energy released in the relaxation events, we find that the energy release time series is intermittent, with large peaks, and the released energy follows an exponential distribution.

Conclusion

We introduced a magnetic topology in the SOC state, on the base of a CA, which is compatible with MHD. The physics it implements is a magnetic field driven by a toroidal current, with resistive instabilities that are relaxed in local diffusive events. The topology as such is not in obvious contradiction with topologies realized in toroidal confinement devices, many aspects and details have though to be worked out, still, and many modifications are possible.

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