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Existence of substantially different solutions in an inverse problem of plasma equilibrium reconstruction

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1. Introduction. An important direction of research in controlled fusion is development of methods for reconstruction of internal plasma parameters using external observations. Such methods allow obtaining valuable information about plasma characteristics, which cannot be measured straightly, understand plasma behavior and produce reliable control technique.

In this paper the problem of toroidal plasma current density reconstruction is considered. It is known that this problem is strongly ill-posed [1-3]. Nevertheless several methods are developed for its solution, for example, those implemented in codes EFIT [4] or SCoPE [2,3], and successfully used for production of practical results.

Typically, methods for current density reconstruction search for some solution of the inverse problem and do not address the question about its uniqueness. However, theoretical analysis of simplified inverse problems reveals possibility of multiple solutions. For correct interpretation of a plasma pulse it is very important to find all substantially different solutions, and then to use supplementary data to select the one appropriate to the real physical process.

In this paper, an accurate formulation of the inverse problem for reconstruction of the poloidal flux and components of the toroidal current density and variations of formulation are presented together with a novel method for determining all substantially different solutions. Examples of substantially different solutions in close to experimental conditions are given.

2. Mathematical formulation of the problem. Let $(\mathrm{R},\eta,\mathrm{Z})$ be cylindrical coordinates with the Z-axis oriented along the tokamak axial symmetry axis; $\psi(\mathrm{R},\mathrm{Z})$ be the poloidal flux function, equal to the covariant component A_η of the vector potential of the magnetic field $\vec{B}=\vec{\nabla}\times\vec{A}$; Γ is known, e.g. from optic measurements [2], plasma boundary in the meridian section of the tokamak; S be the area bounded by Γ; and I be the toroidal plasma current.

The following relations are present in the formulation of the problem of toroidal axially symmetric plasma equilibrium reconstruction

$$R\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial\psi}{\partial R}\right)+\frac{\partial^2\psi}{\partial Z^2}=-\mu_0 R j_\eta,\quad (R,Z)\in S, \tag{1}$$

$$j_\eta = R\frac{dp(\psi)}{d\psi}+\frac{1}{2\mu_0 R}\frac{dF^2(\psi)}{d\psi}, \tag{2}$$

$$\psi\big|_\Gamma = 0, \tag{3}$$

$$\left.\frac{\partial\psi}{\partial\vec{n}}\right|_\Gamma=\Phi,\ \text{where} \tag{4}$$

$$-\frac{1}{\mu_0}\int_\Gamma\frac{\Phi}{R}dl=\int_S j_\eta ds = I\neq 0. \tag{5}$$

The Grad-Shafranov equation (1) is a two-dimensional elliptic equation with non-linear right-hand side. Here the notation of ref. [5] is used: j_η - toroidal current density; $p(\psi)$ - the kinetic pressure; $F(\psi)$ - the poloidal current function, related to the toroidal magnetic field $B_{\mathrm{tor}}=F/R$; $\vec{n}$ - the external in respect to S normal unit vector; Φ - the derivative in the direction of vector $\vec{n}$, which is related to the poloidal magnetic field; μ_0 - the vacuum magnetic permeability coefficient. The total toroidal current I is calculated through Φ with curvilinear integral using the Green formula.

In a typical direct problem function ψ is to be determined from known functions p and F using equations (1)-(3), (5). Below we consider the inverse problem which seeks to find the set (ψ_k, p_k, F_k) of all substantially different solutions, which satisfy conditions (1)-(3), (5) exactly and condition (4) at least approximately with some given inaccuracy δ subject to a given plasma boundary Γ and the function Φ

$$\left\| \left. \frac{\partial \psi_k}{\partial \vec{n}} \right|_{\Gamma} - \Phi \right\| / \|\Phi\| \le \delta \text{, where, for example, } \|\Phi\| \equiv \max_{\Gamma} |\Phi|, \quad \delta \sim 0. \tag{6}$$

In this paper a fast and reliable algorithm for determining of the set of (ψ_k, p_k, F_k) is proposed. For extraction of the unique triplet from substantially different ones (ψ_k, p_k, F_k) it is necessary to add to equations (1)-(6) other constraints appropriate to some additional data.

Partial derivative (4) can be set either from the solution of the direct problem or from experimental data processing. In the last case at least two variants are possible. In the first one the derivative $\partial\psi / \partial\vec{n}$ (4) is searched from the solution of the Grad-Shafranov equation outside plasma using measured at some points values of ψ and $\vec{B}$. In the second one it is assumed that one solution (ψ^*, p^*, F^*), including $\partial\psi^* / \partial\vec{n}$, of an inverse problem, different from (1)-(5), is known, for example, from codes SCoPE or EFIT, which solve for the whole region inside the chamber, and substantially different from (ψ^*, p^*, F^*) solutions (ψ_k, p_k, F_k) are to be found.

Values of $\partial\psi / \partial\vec{n}$ can be found experimentally with some accuracy only. Therefore equation (4) is replaced with a more general condition (6). However equation (5) is preserved, since current I is known from experiment with sufficiently high accuracy.

An important element in the formulation of the problem is setting the class of functions p and F, from which a solution (ψ, p, F) is searched. It is desirable to narrow the class of functions sought for as much as possible. We apply the approach normally used for solution of the direct problem (1)-(3). In the direct problem functions p and F are considered as input parameters and should be preset. The main interest in the direct problem is to find ψ appropriate to the fixed ranges of $p(\psi)$ and $F(\psi)$ values. In this case it is necessary to consider functions $p(\rho(\psi))$ and $F(\rho(\psi))$ [1], in which ρ runs over all given values, e.g. the segment [0,1], for any bounded ψ, otherwise p and F may not get in the required range, since the values of ψ become known only after the solution of the direct problem (1)-(3). Assuming ψ to be bounded and non-negative $\psi \ge 0$ the simplest form of ρ convenient for differentiation is

$$\rho(R,Z) = (\max_{(R,Z)\in S} \psi - \psi) / \max_{(R,Z)\in S} \psi, \quad \rho \in [0,1] \text{ in S.} \tag{7}$$

Thus usually in the direct problem (1)-(3) functions $p(\rho(\psi))$ and $F(\rho(\psi))$ are considered, because their range is known beforehand, since $\rho \in [0,1]$. Geometrical interpretation of transition from $p(\psi)$ and $F(\psi)$ to $p(\rho(\psi))$ and $F(\rho(\psi))$ is transition from setting p and F as function of ψ level lines with unknown in advance numeration to the known in advance numeration of level lines of $\rho(\psi) \in [0,1]$. Substitution (7) does not change the differential properties, but the considered class of functions p and F becomes somewhat narrower, in particular infinitely growing with growth of $|\psi|$ functions are discarded.

Functions p and F in case of $p(\rho(\psi))$ and $F(\rho(\psi))$ become invariant in respect to ψ normalization, i.e. for any constant $C > 0$ we have $p(\rho(\psi)) = p(\rho(C\psi))$ and $F(\rho(\psi)) = F(\rho(C\psi))$. Therefore the free parameter $\lambda \equiv \left(\max_{(R,Z)\in S}\psi - \min_{(R,Z)\in S}\psi\right)^{-2}$ appears in eq. (1) of the direct problem (1)-(3), which is usually chosen from the condition (5) of given total current I.

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[1] Here the new functions $p(\rho(\psi))$ and $F(\rho(\psi))$ are denoted with the same letters p and F, already used for functions $p(\psi)$ and $F(\psi)$, since further it does not lead to collisions of notations.

In the inverse problem by analogy with the direct one we will search functions p and F from the class $p(\rho(\psi))$ and $F(\rho(\psi))$ with $\rho \in [0,1]$ (7), and function ψ from the class of bounded non-negative functions and use condition (5) for ψ normalization. That is we consider the problem (1)-(6) with additional constraint (7). At that transition to ρ in all equations (1)-(6) for example in the boundary conditions, is not necessary, since introduction of ρ is required for defining the right hand side (2) of equation (1) only.

At present the question about uniqueness of the solution of the inverse problem (1)-(5) is studied analytically in some particular cases only. The answer depends on the form of the area S and the form of the right-hand side (2) in eq. (1). It is shown that the problem (1)-(5) can have both one and several solutions. Areas S and functions p, F and Φ, met in practice, require numerical solution of the problem (1)-(5).

3. Numerical method for construction of substantially different solutions. We turn in (2), the right-hand side of eq. (1), to the normalized flux (7) and assume that functions $dp/d\rho$ and $dF^2/d\rho$ can be presented as polynomials in ρ

$$-\mu_0 \frac{dp}{d\rho} = \sum_{i=0}^{m} \alpha_i \rho^i, \quad -\frac{1}{2}\frac{dF^2}{d\rho} = \sum_{i=0}^{m} \beta_i \rho^i \ . \tag{8}$$

The method for determining all substantially different solutions is based on special enumerative technique for values of coefficients α_i and β_i in equation (8). The right-hand side of eq. (1) with fixed coefficients α_i and β_i becomes known and the possibility of solving the direct problem (1)-(3), (5) shows up. We search for the numerical solution by iterations over $s = 1, 2, \ldots$ analogously to [6]

$$\Lambda \psi^{(s)} = \alpha^{(s-1)} \Big(R^2 \sum_{i=0}^{m} \alpha_i (\rho^{(s-1)})^i + \sum_{i=0}^{n} \beta_i (\rho^{(s-1)})^i\Big),$$

$$\alpha^{(s-1)} = -\mu_0 I \Big/ \int_S \Big(R \sum_{i=0}^{m} \alpha_i (\rho^{(s-1)})^i + \frac{1}{R}\sum_{i=0}^{n} \beta_i (\rho^{(s-1)})^i\Big) ds.$$

Here Λ is the difference operator, which approximates the left hand side of eq. (1) taking account of the boundary condition (3). The coefficient α ensures validity of eq. (5) for $\psi = \psi^{(s-1)}$. A modern effective method for solution of this discrete problem is presented in [2], for example. Iterations continue until a steady state or achievement of some maximum number. If the steady state solution exists, we denote it as ψ^{∞}. Inequality (6) is checked for ψ^{∞}. If it is valid then $\psi = \psi^{\infty}$ is taken as the solution if eq. (6). While selecting solutions it is possible to check additional constraints, such as $\psi \geq 0$, or fit to some required pressure p range, etc. with the aim to narrow the set of different solutions (ψ, p, F).

Thus finding all substantially different solutions of the problem (1)-(8) reduces to an accurate overhaul of the values of coefficients in polynomials (8). This can be done by different methods. One is based on usage of the ε-net of the finite number of polynomials [6], which cover a priori defined sufficiently broad class of functions $p(\rho(\psi))$ and $F(\rho(\psi))$ with given accuracy ε. Solutions (ψ_k, p_k, F_k) appropriate to the elements of the ε-net give different solutions of the inverse problem (1)-(8). Using one or other criteria one can select substantially different ones from (ψ_k, p_k, F_k).

4. An example of substantially different solutions. The solution of the direct problem (1)-(3), (5) with MAST-like plasma parameters was considered as the given one (ψ^*, p^*, F^*). The ellipse with elongation 1.7, minor radius 0.5 m and centre at $R =$ 0.7 m was taken for the area S. Total toroidal current was I = 560 kA. Functions $dp^*/d\rho$ and $d(F^*)^2/d\rho$ were calculated with code SCoPE [2,3,5] and presented with polynomials of the 2-d order for $dp^*/d\rho$

and 3-d order for $d(F^*)^2/d\rho$ with $\approx 1\%$ accuracy. Derivative Φ was calculated using ψ^*. The inverse problem (1)-(8) was considered. We searched for its substantially different from (ψ^*, p^*, F^*) solutions in the sense of noticeable deviations in toroidal current density (2).

The method, described briefly in the previous section, gave 198 solutions (ψ_k, p_k, F_k) of the inverse problem (1)-(8), satisfying inequality (6) with accuracy δ < 2.5%. Figs. 1 and 2 illustrate given solution and an obtained one. It is clear that not just quantitative in up to 40% but also qualitative difference in the current density is present. Along with hollow current density the inverse problem has a non-hollow solution. Each of these is appropriate to different physical understanding of the pulse. The fluxes ψ in presented cases differ in less than 5%.

It is important to note that a very similar result was obtained for the inverse problem with fixed plasma pressure $p = p^*$.

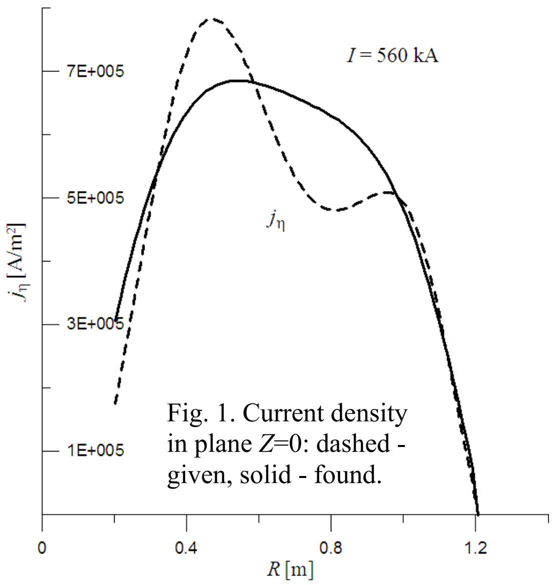

Fig. 1. Current density in plane Z=0: dashed - given, solid - found.

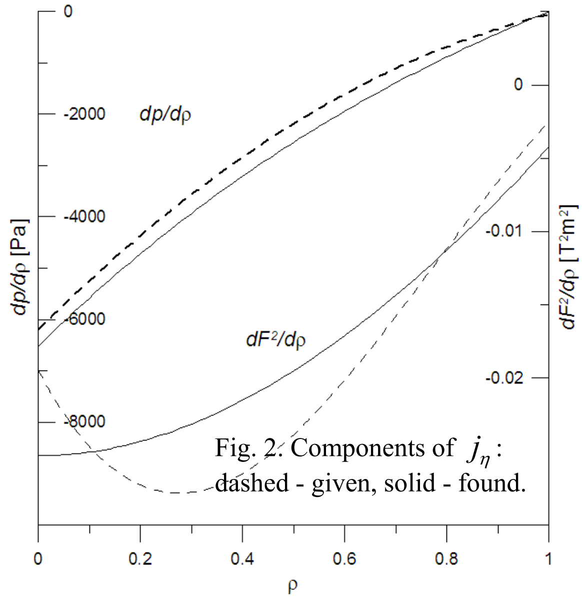

Fig. 2. Components of j_η: dashed - given, solid - found.

5. Conclusion. Formulation of the constraints, required for obtaining a unique solution of the current density, is an unresolved theoretical objective. The possible existence of multiple substantially different solutions, as shown in this paper, indicates the importance of being able to demonstrate uniqueness or alternatively to identify other substantially different solutions. To avoid faulty interpretation of future experiments, caused by possible non-unique reconstructions, it is advisable to supplement equilibrium reconstruction codes with a module for searching of all substantially different solutions. Some previous results may require reexamination for studying the uniqueness of reconstructed current density.

The results of this work confirm a known fundamental statement that finding one solution is not sufficient for some inverse problems, it is essential to explore existence of other substantially different solutions, which also satisfy the conditions of the problem. The presented approach can be used for vindication of uniqueness of the solution or construction of other substantially different solutions for considered and other inverse problems of controlled fusion.

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