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Integral Propagator Solvers for Plasma Kinetic Equations

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Theoretical background

Over a wide class of realistic plasma scenarios, the so-called nonlinear Vlasov-Fokker-Planck equation (VFPE)is used to describe the time evolution of plasma species distribution functions f in the phase space [1, 2, 3]. To solve this equation, numerical computation is required even if any kind of the usual expansions based upon the truncating of a series for f provides a linearized and simplified form of the collision operator. These series truncations may lead to unphysical results, such as negative values of f tails. Consequently, for a feasible kinetic description of the system it would be desirable to directly deal with the full nonlinear nature of the collision operator. Typically, the VFPE reads

$$\frac{\partial f}{\partial t} = -\mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{x}} - \mathbf{a}\cdot\frac{\partial f}{\partial \mathbf{v}} - \frac{\partial}{\partial \mathbf{v}}\cdot\left[\mathbf{D}_v - \frac{\partial}{\partial \mathbf{v}}\cdot\ \mathbf{D}_{vv}\right] f = \mathbf{L} f(\mathbf{v},\mathbf{x},t) \tag{1}$$

where $\mathbf{a}$ is the deterministic acceleration term and $\mathbf{D_v}$ and $\mathbf{D}_{vv}$ are the drift vector and the diffusion tensor. The so-called Vlasov term $\mathbf{v}\cdot\partial/\partial\mathbf{x}$ accounts with spatial plasma inhomogeneities. This equation is a particular case of the general nonhomogeneous Fokker-Planck equation [4] given by

$$\frac{\partial f}{\partial t} = \mathbf{L}_{FP}(\mathbf{q},t) f(\mathbf{q},t) + \rho(\mathbf{q},t) = -\frac{\partial}{\partial \mathbf{q}}\cdot\left[\mathtt{A}\ (\mathbf{q},t) - \frac{\partial}{\partial \mathbf{q}}\cdot\mathbf{D}_{qq}(\mathbf{q},t)\right] f + \rho(\mathbf{q},t), \tag{2}$$

where $\mathbf{q}$ means the vector $(\mathbf{v},\mathbf{x})$. The drift vector $\mathtt{A}\ = (\mathtt{A}_{\mathbf{v}},\mathtt{A}_{\mathbf{x}}) = (\mathbf{a}+\mathbf{D}_v,\mathbf{v})$ and the symmetric diffusion tensor $\mathbf{D}_{qq}$, as well as the source-sink term ρ, may also depend on f through a nonlinear integro-differential relation, as it happens with the plasma FPE in Landau form. The formal solution of (2) can be expressed as

$$f(\mathbf{q},t) = \int f(\mathbf{q}',t')\Pi(\mathbf{q},t \mid \mathbf{q}',t')d\mathbf{q}' + \int d\mathbf{q}' \int_{t'}^{t} dT\ \rho(\mathbf{q}',T)\Pi(\mathbf{q},t \mid \mathbf{q}',T) \tag{3}$$

if a propagator $\Pi(\mathbf{q},t \mid \mathbf{q}',t')$ (satisfying the same equation with initial Dirac delta function) were known. For Markovian processes, Π has the sense of a conditional probability transition form point $\mathbf{q}'$ at time t' to point $\mathbf{q}$ at time $t > t'$. This interpretation may be held for the nonlinear VFPE when use is made of an approximate (nonunique) propagator $P_\tau \approx \Pi$ for small values of $\tau = t - t'$, which is the multi-variate Gaussian distribution [4, 5, 6] $P_\tau(\ \mathbf{v},\ \mathbf{x};\ \mathbf{v}',\ \mathbf{x}'|t) = \exp\left[-(\mathbf{q}-\overline{\mathbf{q}})\cdot(\sigma^2)^{-1}\cdot(\mathbf{q}-\overline{\mathbf{q}})/2\right] / \left[\sqrt{\|(\sigma^2)\|}\ (2\pi)^N\right]$ with mean $\overline{\mathbf{q}} = \mathbf{q}' + \tau\mathtt{A}\ '$ and covariance $(\sigma)^2 = 2\tau\mathbf{D}'_{vv}$ if the diffusion tensor is invertible ($N = 3$ for the $(3+3)$-D VFPE) [4]. Primed functions are computed in the source variables $\mathbf{q}'$, rather than of at the field points $\mathbf{q}$. Because of $\mathbf{D}_{xv} = 0$ and $\mathbf{D}_{xx} = 0$, $\|\mathbf{D}_{qq}\| = 0$ for (1) this P_τ is no longer valid. Nevertheless, other short-time propagators can be obtained by constructing an auxiliary solvable VFPE. In particular, following [6], using the Dirac delta function property (as an operator) $g(\mathbf{q})\delta(\mathbf{q}-\mathbf{q}') = g(\mathbf{q}')\delta(\mathbf{q}-\mathbf{q}')$, applied to $\mathbf{A} = \mathbf{D}_v + \mathbf{a}$ and $\mathbf{D}_{ij} = D_{vi\,vj}$, the simplest P_τ is again a Gaussian distribution with mean $\overline{\mathbf{q}}$ and nonsingular covariance tensor (σ^2) given by

$$\overline{\mathbf{q}} = (\overline{\mathbf{v}}\,,\,\overline{\mathbf{x}}) = (\mathbf{v}' + \mathbf{A}'\tau\,,\,\mathbf{x}' + \mathbf{v}'\tau + \frac{1}{2}\mathbf{A}'\tau^2)\,,\;(\sigma^2) = \begin{pmatrix} (\sigma^2)_{\mathbf{vv}} & (\sigma^2)_{\mathbf{vx}} \\ (\sigma^2)_{\mathbf{xv}} & (\sigma^2)_{\mathbf{xx}} \end{pmatrix} = 2\tau \begin{pmatrix} \mathbf{D}'_{vv} & \tau\mathbf{D}'_{vv}/2 \\ \tau\mathbf{D}'_{vv}/2 & \tau^2\mathbf{D}'_{vv}/3 \end{pmatrix}. \quad (4)$$

From (3), using P_τ instead of Π, a stable explicit and robust grid-free numerical scheme is provided for advancing f in time. Only the (very few) non zero P_τ numerical values contribute to the computation. For spatially uniform problems, as well as if f is homogenous with respect to one or several variables, a simplified propagator $P_\tau(\mathbf{v};\mathbf{v}'|t)$ can be obtained by integrating P_τ over the irrelevant variables.

Otherwise, nonnatural boundary conditions may be incorporated to the integral scheme as a surface integral, involving only P_τ and its derivatives for unbounded space as in [5].

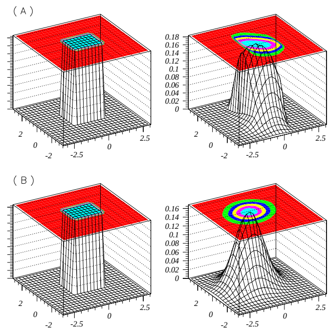

Figure 1: f spatial diffusion for finite-difference (A) and integral schemes (B)

Applications. Local Transport. The form of P_τ found for (1) clearly renders how xv cross-diffusion and xx self-diffusion processes are driven by diffusion coefficients proportional to τ and τ^2 and $\mathbf{D}_{vv}$, which also accounts with collisional effects as well as other ones as inverse bremsstrahlung absorption and RF-heating. Note that the diffusive behavior is not explicitly depicted by the differential VFPE. For this reason, a finite-difference numerical scheme, or the so-called splitting methods, could rise abnormal description for the f evolution, as shown in the first row of Fig. 1. On the other hand, drifting processes are pictured by $\overline{\mathbf{q}}$, which obeys the deterministic motion law of a particle in phase space for short τ. This suggests that $\overline{\mathbf{q}}$ in P_τ could be replaced by its deterministic motion equation, combining in this way the advance scheme for f with deterministic or other simulation methods connecting deterministic and stochastic (Langevin) processes [6, 7].

To make more tractable the general six-fold VFPE in connection to transport calculations, further simplifications are required [3, 8, 9]. For instance, the Vlasov term, as being the divergence of $\mathbf{v}f$, can be locally modelized by an effective source-sink ρ through a flux particle balance in a thin slab of plasma. This leads to a nonhomogeneous FPE for the electron distribution function f_e that can be cast into the form of (2) whereas (3) specializes into the simpler form $f_e(\mathbf{v},t+\tau) = \int [f_e(\mathbf{v}',t) + \tau\rho(\mathbf{v}',t)]P_\tau(\mathbf{v};\mathbf{v}'|t)\, d\mathbf{v}'$.

The nonlinearized collisional Fokker-Planck equation in the Landau form, $C(f_e, f_e) = \mathbf{L}_{FP} f_e$, has the drift and diffusion coefficients $D^e = D^{e/e} + D^{e/i}$ computed through the anisotropic Rosenbluth's potentials for a fully ionized plasma with massive ions at rest [2, 10]. Using cylindrical geometry for $\mathbf{v} = (v_\parallel, v_\perp)$ with the parallel component along the temperature gradient (or the electric field $\mathbf{E}$ on the Z axis. The shot-time propagator, given in [11] in detail, automatically preserves the electron ensemble number density and momentum at any time step whereas the energy loses (of order τ^3) are controlled and reduced by using a numerical self-control algorithm that performs P_τ by adding corrections terms of order τ^3 in its mean or covariance elements. The plasma slab considered here has the width of n_c times the mean electron-ion collision length l_{ei}, $\Delta Z = n_c l_{ei}$. As a function of the left (l) and right (r) particle fluxes $\mathbf{\Phi}$ the effective local source is

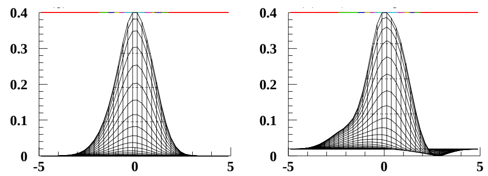

Figure 2: f_e and f_{spt} profiles.

$$\rho = \rho(f_e, \mathbf{v}) = \Delta\Phi/\Delta Z = [v_\parallel^+ (f_l - f_e) - v_\parallel^- (f_e - f_r)]/\Delta Z \qquad (5)$$

where $v_\parallel^\pm = (v_\parallel \pm |v_\parallel|)/2$ and $\mathbf{\Phi}$ is the particle flux per unit area through the left and right surfaces at $z = \mp\Delta Z/2$. The left and right distributions $f_{l,r}$ are assumed to be Maxwellian f_e^0 with densities $n_{l,r}$ and temperatures $T_{l,r}$ computed to balance particle and momentum fluxes over the slab, in order to compare with classical transport results. In our calculations the plain form of the collision operator is used with the electric field added to the $D_\parallel^e$ electron drift coefficient. The Spitzer and Harm's transport coefficients are recovered for vanishingly small generalized thermodynamical forces. This is owed to the fact of using Maxwellian functions at both sides of the slab, which is , in fact, not necessary. This turns ρ in (5) into the well-known perturbative term $f_e^0(mv^2/2kT - 5/2)\mathbf{v}\cdot\partial \ln T/\partial\mathbf{r}$ used for classical transport calculations for the zero limit fields. Thereupon, for a fully coincidence with classical fluxes, smooth temperature gradients are required in a characteristic scale length of about $100 l_{ei}$. On the contrary, by also increasing E and the temperature gradients with ΔZ comparable to l_{ei}, the electric current density J and the heat flux Q (Fig. 3) undergo drastic reductions, up to 65% with respect the classical linear transport results. For any finite fields, the first order approximate distribution $f_e = f_{spt}$ for linear classical calculus develops negative amplitudes in the range of high velocities, whereas our f_e remains positive for all $\mathbf{v}$ at any time, having plain physical sense. The comparison is shown in Fig. 2.

Conclusions and prospects. By combining analytical and computational efforts, a short-time Green's function P_τ brings forth a way to implement f in time through a physically meaningful path integral method, which preserves the positiveness, smoothness and the norm of f. This solution procedure would allows us to deal with the VFPE coupled to Maxwell equations to self-consistently describing the nonlinear plasma dynamics by considering f (or P_τ) as an exact distribution from which the information to solve the electromagnetic equations under the usual potential formalism can be extracted. Furthermore, the method could be used to nonlocal transport calculations accounting with the nonlinear nature of the Fokker-Planck-Landau operator. To local transport investigation it is enough to model the Vlasov term via the effective source-sink (5) and applying the elementary velocity dependent P_τ .Extension of the method to other collisional plasma operators is a topic for future research.

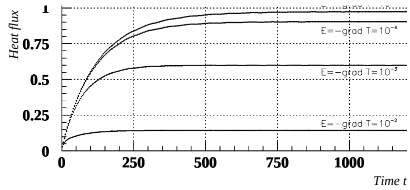

Figure 3: Heat flux inhibition relative to classical results

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