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Analytic Magnetohydrodynamic Equilibria of a Magnetically Confined Plasma with Incompressible Flows

Ch. Simintzis, G. N. Throumoulopoulos, G. Pantis

University of Ioannina, Association Euratom - Hellenic Republic, Section of Theoretical Physics, GR 451 10 Ioannina, Greece

H. Tasso

Max-Planck-Institut für Plasmaphysik, Euratom Association, D-85748 Garching, Germany

1. Introduction and conclusions.

There has been established in a variety of magnetic configurations that sheared flows can reduce turbulence and produce transport barriers which under certain conditions can extend up to the whole plasma [1]. In view of a fusion reactor the spontaneous formation of transport barriers, i.e. those driven by internal processes even in the absence of a momentum source, is of particular interest. In an attempt to contribute to understanding the confinement properties of fusion plasmas we considered magnetohydrodynamic equilibria with incompressible flows in cylindrical [2, 3] and toroidal [4] geometries. For a toroidal plasma we found that the equilibrium is governed by a second-order elliptic differential equation for the poloidal magnetic flux function ψ containing five surface quantities along with an algebraic relation for the pressure, which are amenable to analytic solutions. Analytic stationary equilibria with constant poloidal Mach-numbers M_p^2 (defined in Sec. 2) were also obtained.

In the present work we construct analytic equilibria with $M_p^2 \neq$ const. associated with monotonically increasing safety-factor profiles and differentially varying "radial" (perpendicular to magnetic surfaces) electric fields $\mathbf{E}$ similar to those observed during the transition from the low-confinement mode to the high-confinement mode in tokamaks. It turns out that the flow affects certain confinement figures of merit, e.g. the shape of the magnetic surfaces, the toroidal beta β_t, the safety factor q, and the magnetic shear s. Most importantly, it is $\mathbf{E}$ and its "radial" variation which impacts equilibrium. The conclusions on cylindrical and toroidal configurations are summarized in Sections 2 and 3, respectively. The study will be reported in detail elsewhere.

2. Cylindrical equilibria [1]

The equilibrium of a cylindrically symmetric plasma with incompressible flow and arbitrary cross-sectional shape satisfies (in convenient units) the differential equation for ψ [3]

$$\left[1 - \frac{(F')^2}{\rho}\right] \nabla^2\psi - \frac{1}{2}\left(\frac{(F')^2}{\rho}\right)' |\nabla\psi|^2 + \left(P_s + \frac{B_z^2}{2}\right)' = 0, \qquad (1)$$

stemming from the "radial" component of the force-balance equation $\rho(\mathbf{v}\cdot\nabla)\mathbf{v} = \mathbf{j}\times\mathbf{B} - \nabla P$. Here, $P_s(\psi)$, $\rho(\psi)$, and $B_z(\psi)$ are, respectively, the static pressure, density,

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and magnetic field parallel to the cylindrical axis z, which remain constant on magnetic surfaces ψ = const.; the surface quantity $F(\psi)$ is related to the poloidal flow; the prime denotes differentiation with respect to ψ. Because of symmetry the equilibrium quantities are z-independent and the axial velocity v_z does not appear explicitly in Eq. (1). The magnetic field, current density, plasma velocity, and electric field associated with the flow can, respectively, be expressed as $\mathbf{B} = B_z\nabla z + \nabla z \times \nabla\psi$, $\mathbf{j} = \nabla^2\psi\nabla z - \nabla z \times \nabla B_z$, $\rho\mathbf{v} = \rho v_z(\psi)\nabla z + \nabla z \times \nabla F$, and $\mathbf{E} = -\nabla\Phi(\psi)$. The component of Ohm's low $\mathbf{E}+\mathbf{v}\times\mathbf{B} = 0$ yields the relation $(B_z F')/\rho - v_z = \Phi'$ and therefore five out of the surface quantities P_s, ρ, B_z, F, v_z and Φ' remain arbitrary. Unlike to the case of static equilibria the pressure $P = P_s(\psi) - (1/2)M_p^2(\psi)|\nabla\psi|^2$ is not a surface quantity. ($M_p^2 = (F')^2/\rho$ is the Mach-number of the poloidal velocity with respect to the poloidal-magnetic-field Alfvén velocity). Also, the axial current density $j_z = (1 - M_p^2)^{-1}\left[(1/2)(M_p^2)'|\nabla\psi|^2 - (P_s + B_z^2/2)'\right]$ is affected by M_p^2 and its "radial" variation.

Under the transformation

$$u(\psi) = \int_0^\psi [1 - M_p^2(g)^{1/2}]\, dg \quad M_p^2 < 1, \tag{2}$$

Eq. (1) reduces [after dividing by $(1 - M_p^2)^{1/2}$] to

$$\nabla^2 u + \frac{d}{du}\left(P_s + \frac{B_z^2}{2}\right) = 0. \tag{3}$$

Eq. (3), free of the nonlinear term $1/2(M_p^2)'|\nabla\psi|^2$ and identical in form with the static equilibrium equation, is analytically solvable for sheared flows with $M_p^2 < 1$. We have constructed solutions for (i) flat static axial current density profiles [$(d/du)\,(P_s + B_z^2/2)$ = const.] and (ii) peaked axial current density profiles [$(d/du)\,(P_s + B_z^2/2) \propto u$), either non-vanishing on the plasma boundary ($M_p^2 \propto u$) or vanishing on the plasma boundary ($M_p^2 \propto u^2$) (Hereinafter the values of any quantity on the magnetic axis and on the plasma boundary will be denoted by the subscripts a and b, respectively; the subscript s indicates a static-equilibrium quantity). In addition, configurations with circular and elliptical cross-section boundaries have been considered. As examples, we give here the simple solutions $u_i(x, y) = u_a\,(1 - x^2/\mathcal{A}^2 - y^2/\mathcal{B}^2)$ and $u_{ii}(r) = u_a J_0(2.4r/r_b)$, describing, respectively, elliptic configurations of case (i) and circular configurations of case (ii); (x, y) are Cartesian coordinates; the ellipse $x^2/\mathcal{A}^2 + y^2/\mathcal{B}^2 = 1$ determines the boundary cross-section; r is the radial distance from the z-axis; and J_0 is the zeroth-order Bessel function. On the basis of the solutions obtained we came to the following conclusions.

1. The "toroidal" beta $\beta_z \equiv 2 < P > /B_{zb}^2$ takes lower values as compared with the static-equilibrium ones and decreases as M_p^2 increases.

2. Depending on the direction and the magnitude of the flow and its "radial" variation as well as on the location of a magnetic surface, the safety factor q can take either higher or lower values as compared with the static-equilibrium ones.

3. In the presence of flow the magnetic shear s increases.

4. The electric field vanishes on the magnetic axis and on the boundary, and it has an extremum close to the boundary. Its magnitude becomes larger as flow increases.

5. For an elliptical boundary β_t, q, s and the magnitude of $\mathbf{E}$ become larger as compared with the circular-boundary ones with the same area, and they increase as the elongation of the boundary becomes larger.

It should be noted, however, that for tokamak flows, for which $M_p^2 \approx M_z^2 \equiv v_z^2/(B_z^2/\rho) \approx 0.1$, the variation of any equilibrium quantity A due to the flow is found small, i.e. $|(A - A_s)/A_s| < 0.1$. Although such small variations could contribute to internal processes (at present incompletely understood) producing (spontaneously) transport barriers, the above mentioned results indicate that the most important impact of the flow should be related to $\mathbf{E}$.

3. Toroidal equilibria [2]

For an axisymmetric toroidal system it is convenient to use cylindrical coordinates (R, ϕ, z) with z corresponding to the axis of symmetry. The equilibrium quantities do not depend on ϕ. Owing to the toroidicity associated with $|\nabla\phi| = 1/R$ the equilibrium equation for ψ assumes the form [4]

$$(1 - M_p^2)\Delta^\star\psi - \frac{1}{2}(M_p^2)'|\nabla\psi|^2 + \frac{1}{2}\left(\frac{X^2}{1 - M_p^2}\right)' + R^2\,(P_s)' + \frac{R^4}{2}\left(\frac{\rho(\Phi')^2}{1 - M_p^2}\right)' = 0, \qquad (4)$$

the surface quantity $X(\psi)$ being related to the toroidal magnetic field and $\Delta^\star \equiv R^2\nabla\cdot(\nabla/R^2)$. Note the R^4-flow term depending on $\mathbf{E}$ and its "radial" variation. For vanishing flow Eq. (4) reduces to the Grad-Schlüter-Shafranov equation. The magnetic field, current density, and velocity are now given by $\mathbf{B} = I(R,z)\nabla\phi + \nabla\phi\times\nabla\psi$, $\mathbf{j} = \Delta^\star\psi\nabla\phi - \nabla\phi\times\nabla I$, and $\rho\mathbf{v} = \Theta(R,z)\nabla\phi + \nabla\phi\times\nabla F(\psi)$. On the basis of the "radial" component of Ohm's law and the $\nabla\phi$-component of the force-balance equation the toroidal quantities I and Θ can be expressed in terms of R and surface quantities as: $I(\psi, R) = (X - R^2\Phi'F')/(1 - M_p^2)$ and $\Theta(\psi, R) = (XF' - R^2\rho\Phi')/(1 - M_p^2)$. Also, the relation for the pressure is $P = P_s(\psi) + \rho\left[v^2/2 + \left(R^2(\Phi')^2\right)/(1 - M_p^2)\right]$.

Under transformation (2) Eq. (4) is put in the form

$$\Delta^\star u + \frac{1}{2}\frac{d}{du}\left(\frac{X^2}{1 - M_p^2}\right) + R^2\frac{dP_s}{du} + \frac{R^4}{2}\frac{d}{du}\left(\rho\frac{d\Phi}{du}\right)^2 = 0. \qquad (5)$$

Eq. (5) can be linearized and solved for several physically reasonable choices of $X(u)$, $M_p(u)$, $\rho(u)$, $P_s(u)$, and $\Phi(u)$. As an example, we will obtain a solution by means of the ansatz $(d/du)P_s = -P_{sa}/u_b$, $(d/2du)[\rho(d\Phi/du)^2] = -\lambda(P_{sa})/[u_b(1+\delta^2)R_0^2]$, $(d/2du)[X^2/(1 - M_p^2)] = \gamma P_{sa}/[u_b(1+\delta^2)]$, $\rho = \rho_a(1 - u/u_b)^\kappa$ $(0 < \kappa < 1)$, and $M_p^2 = M_{pa}(1 - u/u_b)^\mu$ $(\mu > 1)$. Here, λ is a flow parameter; γ is related to the magnetic properties

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of the plasma, i.e. for $\gamma > 0$ the plasma is diamagnetic; δ is related to the shape of the magnetic surfaces; and κ and μ are necessary for P to vanish on the boundary and for the poloidal current density to be tangential to the boundary. The solution of Eq. (5) then is

$$u = \left[z^2(R^2-\gamma) + \frac{\delta^2+\lambda}{4}(R^2-R_0^2)^2 + \frac{\lambda}{12R_0^2}(R^2-R_0^2)^3 \right] \frac{P_{sa}}{2u_b(1+\delta^2)}. \qquad (6)$$

For vanishing flow Eq. (6) reduces to the static equilibrium [5] which has been extensively employed in tokamak confinement studies. Owing to the flow, configurations with two magnetic axes ($\lambda < 0$) in addition to the usual ones with a single magnetic axis ($\lambda > 0$) are possible. The primary characteristics of the configurations with a single magnetic axis are the following.

1. The magnetic axis is located on ($z = 0, R = R_0$) where $u = 0$ and, depending on the position of the boundary, the plasma can extent to a separatrix where u assumes its maximum value. The inner part of the separatrix, close to the axis of symmetry, is defined by the vertical line $R = \sqrt{\gamma}$ and, therefore, for $\gamma = 0$ Eq. (6) can describe a compact toroidal equilibrium. The outer part of the separatrix has finite triangularity, unlike the case of static equilibria for which it is elliptical.

2. The magnetic surfaces in the vicinity of the magnetic axis are ellipses with semi-axes given by

$$a_z = \sqrt{u}\left[\frac{2(1+\delta^2)u_b}{P_{sa}(R_0^2-\gamma)}\right]^{1/2} \quad \text{and} \quad a_R = \frac{\sqrt{u}}{R_0}\left[\frac{2(1+\delta^2)u_b}{P_{sa}(\delta^2+\lambda)}\right]^{1/2}.$$

   The ellipses are more elongated parallel to the z-axis as compared with the static-equilibrium ones, i.e. $a_z/a_R = (1+\lambda/\delta^2)^{1/2}(a_z/a_R)_s$, where $(a_z/a_R)_s = \delta R_0/\sqrt{R_0^2-\gamma}$.

Also, analytic evaluation of q_a, and $\mathbf{E}$ on the middle-plane $z = 0$ leads to conclusions consistent with those reported in Sec. 2 for cylindrical equilibria.

Finally, we note that it is interesting to extend the present study to equilibria with reversed magnetic shear usually related to internal transport barriers.

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