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Field-Reversed Configuration (FRC) As A Minimum-Dissipative Relaxed State

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Abstract: It is well known that Field-reversed configurations (FRC) show remarkable stability on the MHD time scale, but not much attempts have been made to see FRC as a relaxed state. FRC, being a non force-free state, is not attained through a Taylor type relaxation. In this work, we attempt to establish, for the first time, that FRC can be viewed as a relaxed state with minimum dissipation and constant energy. The Euler-Lagrange equation describing such relaxed states is solved by superposing the solutions of the force-free equation, with the boundary conditions representing zero current density at the edge of the plasma. While one choice of the eigenvalues of the superposing Taylor-states leads to a FRC, with a completely null toroidal field and high plasma beta, the other choice exhibits the spheromak, thus showing two distinct branches of the solution.

Introduction: The field-reversed configuration (FRC) is a compact toroidal confinement device [1] without a toroidal magnetic field. The equilibrium consists of closed poloidal flux surfaces, distinguished from the open flux surfaces by a separatrix. The plasma pressure is maximum at the magnetic vortex point where the poloidal field goes to zero.

Analytical equilibria with elongated flux surfaces describing the FRC have been constructed using the Hill's vortex model. Such a description is only appropriate for regions inside the separatrix. Several families of analytical equilibria [2-4] have improved upon the Hill's vortex type solutions in the interior and exterion regions bringing the theoretical results closer to experimentally observed situtaions.

Recent experimental investigations [5] as well as numerical simulations [6] on merging of two spheromaks with the subsequent formation of a high beta FRC with zero toroidal field or a new low beta spheromak with toroidal magnetic field comparable to the poloidal field are suggestive of a relaxation phenomena that can lead to both these two branches. The gross features of the relaxed states of a spheromak are adequately described by Taylor's force-free equation [6] of $\nabla \times \mathbf{B} = \lambda \mathbf{B}$. The relaxation processes involved in the formation of FRC that is accompanied by a total annihilation of magnetic helicity cannot be explained by Taylor's theory that is based on the principle of minimum magnetic energy for an invariant magnetic helicity. Moreover, FRC is known to support finite pressure gradients as well as non zero perpendicular component of current that further demonstrate its non force-free characteristics. Theoretical analysis on relaxed states of FRC have earlier been obtained by Hameiri and Hammer [7] by postulating the principle of maximum entropy and more recently by Steinhauer [8] by minimizing the total magnetofluid energy. The physical processes through which a non force-free state emerges from the fusion of two Taylor states strongly calls for further investigations.

In this work we propose to study the FRC as a relaxed state obtained from the principle of minimum dissipation. The principle of minimum dissipation was first utilized by Montgomery and Phillips [9] to understand the steady state profiles of RFP configuration under the constraint of a constant rate of supply and dissipation of helicity. It has been shown [11] that these classes of relaxed states can support a non-zero pressure gradient together with a field reversal for reversed field pinch devices. As the FRC formation is accompanied by a dissipation of magnetic energy [5] and an annihilation of magnetic helicity while approximately conserving the plasma energy, we try to study the characteristic features from a variational principle based on minimum dissipation rate and constant energy.

Euler-Lagrange Equation: The variational equation is obtained as

$$\delta \int (\eta \mathbf{j}^2 + \overline{\lambda} \mathbf{B}^2) dV = 0 \tag{1}$$

where $\overline{\lambda}$ is the Lagrange's undetermined multiplier. This leads to the Euler-Lagrange equation

$$\nabla \times \nabla \times \mathbf{B} = \Lambda \mathbf{B} \tag{2}$$

where $\Lambda = \overline{\lambda}/\eta$. The above equation was earlier discussed by Chandrasekhar and Woltjer [10] in a sequel to the complete general solution of the force-free equation in terms of the Chandresekhar and Kendall (CK) [12] eigenfunctions.

FRC and Spheromak Solutions: The solution of the above equation can be constructed as a linear combination of the solutions of the Taylor's force-free equation

$$\mathbf{B} = \sum_{i=1,2} \alpha_i \mathbf{B}_i \quad \text{with} \quad \nabla \times \mathbf{B}_i = \lambda_i \mathbf{B}_i \tag{3}$$

Here α_i are constants to be fixed by boundary conditions.

It can be shown that the expression for $\mathbf{B}$ in eq. (3) satifies eq. (2) when (i) $\lambda_1 = \lambda_2 = \sqrt{\Lambda}$. (ii) $\lambda_1 = -\lambda_2 = \sqrt{\Lambda}$. While the first choice leads to $\mathbf{B}$ satisfying the force-free equation, the second choice brings forth a set of solutions of eq. (2) that do not satisfy the force-free equation. Since the Rosenbluth and Bussac's spheromak solutions [13] corresponding to the first choice are well known, in the following, we will work with the second choice, i.e., $\lambda_1 = -\lambda_2$ in order to demonstrate the FRC type solutions of eq. (2).

The CK eigenfunctions are given by

$$\mathbf{B}_i = \mathbf{r} \times \nabla \psi_i + \frac{1}{\lambda_i} \nabla \times (\mathbf{r} \times \nabla \psi_i) \tag{4}$$

where ψ_i are solutions of $(\nabla^2 + \lambda_i^2)\psi_i = 0$ and $\mathbf{r}$ is the position vector. In spherical polar coordinates ψ_i are obtained for the axisymmetric case (toroidal symmetry) with $m = 1$ (where m is the poloidal mode number) as

$$\psi_i = j_1(\lambda_i r)\cos\theta \tag{5}$$

.

The corresponding magnetic fields are obtained as

$$B_{1r} = B_{2r} = -2\frac{j_1(\sqrt{\Lambda}r)}{\sqrt{\Lambda}r}\cos\theta$$

$$B_{1\theta} = B_{2\theta} = \left(\frac{j_1(\sqrt{\Lambda}r)}{\sqrt{\Lambda}r} + j_1'(\sqrt{\Lambda}r)\right)\sin\theta \tag{6}$$

$$B_{1\phi} = -B_{2\phi} = -j_1(\sqrt{\Lambda}r)\sin\theta$$

The helicities in these states are obtained from

$$K_1 = \int \mathbf{A}_1\cdot\mathbf{B}_1 dV, \quad K_2 = \int \mathbf{A}_2\cdot\mathbf{B}_2 dV$$

and the resultant helicity in the states represented by eq. (2) is obtained as the superposition of the individual helicities

$$K = \alpha_1 K_1 + \alpha_2 K_2$$

It can be easily shown that for choice (i) $K_1 = K_2$ and for (ii) $K_1 = -K_2$. Also, the states characterized by $\mathbf{B}_1$ and $\mathbf{B}_2$ have plasma $\beta = 0$. In order to fix the eigenvalue Λ as well as the constants α_1 and α_2 we consider the boundary condition $\mathbf{j} = 0$ at the plasma edge defined by $r = a$. This leads to the following conditions

$$\alpha_1 = \alpha_2$$

$$j_1(\sqrt{\Lambda}a) = 0 \rightarrow \sqrt{\Lambda}a = 4.493 \tag{7}$$

From eqs. (3) and (6) it is clear that this leads to $B_\phi = 0$ as is appropriate for a FRC. These relaxed states support a perpendicular component of component of current as the choice of boundary conditions leads to $j_\phi \neq 0$. This demonstrates a distinct non force-free character of these relaxed states supporting a finite pressure gradient. The resultant helicity in this state obtained from the choice (ii) is zero as $K_1 = -K_2$. Although the states characterized by $\mathbf{B}_1$ and $\mathbf{B}_2$ have plasma $\beta = 0$, the new relaxed state has an average β value as high as 3.0. This high value of β together with the vanishing of toroidal magnetic field are indicative of the FRC nature of these solutions.

For the choice (i) it is obvious that the resultant state, with $\lambda_1 = \lambda_2$ is a Taylor state with zero beta. In this branch of solutions of eq. (2) because $K_1 = K_2$, the helicity is enhanced and also B_ϕ is nonzero as $B_{1\phi} = B_{2\phi}$. This state with the eigenvalue given by 4.493 represents the standard spheromak configuration of Rosenbluth and Bussac [13] .

Conclusions: This work demonstrates the characterisitics of two distinctly different kinds of solutions of the double-curl Euler-Lagrange equation obtained from the principle of minimum dissipation and constant energy. For a given choice of the boundary condition taken as $\mathbf{j} = 0$ at the edge of the plasma, one set of solutions demonstrates the emergence of FRC as a high beta, zero helicity non force-free state from the superposition of two Taylor states with eigenvalues that are equal in magnitude but opposite in sign. The other branch of solution of the equation exhibits force-free characteristics with zero beta and non zero helicity and can be interpreted as being close to a spheromak type configuration. Further investigations with the inclusion of flow energy in the model so as to conserve the total plasma energy instead of only magnetic energy will lead to more critical understanding of the reason behind the bifurcation to such distinctly different branches as spheromak and field reversed configuration. Also an analysis of the problem in geometries other than spherical, such as prolate or oblate geometries will bring the results closer to experimentally observed FRC configurations.

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