Dissipative Magnetohydrodynamic Equilibria with Compressible Fluid Flow

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Introduction. Modeling the dissipative equilibrium state of a toroidally confined, axisymmetric plasma as that of a compressible Newtonian fluid with scalar viscosity, resistivity and thermal conductivity we derive a closed set of partial differential equations for eight scalar functions describing the magnetohydrodynamic steady state. Implicitly taking into account an equation of state and $\nabla\cdot\mathbf{B} = 0$, these functions embody the pressure p, the temperature T, the electric potential Φ, the velocity stream function S_M forced by the expansion rate $\nabla\cdot\mathbf{v} \neq 0$ and the poloidal fluxes (Ψ, Ψ_M) and toroidal circulations (C, C_M) of magnetic induction $\mathbf{B}$ and mass velocity $\mathbf{v}$. This set follows from an extended analysis [1] of stationary nonideal MHD equations for momentum balance including inertia, Ohm's law and the conservation of mass and energy. The resultant Poisson problems involve Laplace (Δ) and Stokes $(\Delta^* = R^2\nabla\cdot(R^{-2}\nabla))$ operators: Δ acting on p, T, S_M, and Φ, and Δ^* on Ψ, Ψ_M, C and C_M.

We give a conclusive derivation of the equations and discuss suitable boundary conditions to be imposed. Also an algorithm for their efficient solution is described and results of its application are presented.

MHD Background. Momentum balance and Ohm's law are assumed to be of the form

$$\rho\mathbf{v}\cdot\nabla\mathbf{v} = \nabla\times\mathbf{B}\times\mathbf{B}/\mu_0 - \nabla p - \nabla\cdot\Pi \tag{1}$$

$$\mathbf{v}\times\mathbf{B} = \nabla\Phi + U\nabla\phi + \eta\nabla\times\mathbf{B}/\mu_0 \tag{2}$$

Φ is the electric potential, U the loop voltage of an externally applied toroidal electric field. Π is the deviatoric part of the pressure tensor with

$$\Pi \simeq -2\mu\mathbf{D} \qquad \mathbf{D} = \frac{1}{2}\left\{\nabla\mathbf{v} + (\nabla\mathbf{v})^T\right\} - \frac{1}{3}\mathbf{I}\,\nabla\cdot\mathbf{v} \tag{3}$$

where μ is the scalar viscosity and $\mathbf{D}$ the deformation tensor for a compressible fluid.

The Effect of Scaling Dissipation. For a discussion of the effects of scaling the dynamic viscosity μ and plasma resistivity η we temporarily nondimensionalize the equations (1) and (2) by introduction of a macroscopic scale length a, and characteristic values for mass density, pressure and flow velocity. Then, with the viscous and resistive Lundquist numbers

$$\mathcal{L}_\mu = a(p_o\rho_o)^{1/2}/(\mu M), \quad \mathcal{L}_\eta = \mu_0 a(p_o/\rho_o)^{1/2}M/\eta \tag{4}$$

where $M = v_o/(p_o/\rho_o)^{1/2}$ is the Mach number, momentum balance and Ohm's law can be written in dimensionless form as

$$\mathcal{L}_\mu(M^2\rho\mathbf{v}\cdot\nabla\mathbf{v} - \nabla\times\mathbf{B}\times\mathbf{B} + \nabla p) + 2\nabla\cdot\mathbf{D} = 0 \tag{5}$$

$$\mathcal{L}_\eta(\nabla\Phi - \mathbf{v}\times\mathbf{B}) + \nabla\times\mathbf{B} = 0 \tag{6}$$

The limit $\mathcal{L}_\mu = \mathcal{L}_\eta = \infty$ describes ideal, in general compressible flow equilibria. Here, for poloidal Mach numbers $M_p \sim v_p/c_s < B_p/B$ (c_s – speed of sound), (5) leads to an elliptic boundary value problem for the poloidal magnetic flux Ψ [2], i.e. for the magnetic induction, whereas (6) determines the class of solution compatible velocity fields algebraically. On the contrary, in the extreme limit $\mathcal{L}_\mu = \mathcal{L}_\eta = 0$, equation (5) must be considered as responsible differential equation for $\mathbf{v}$ and (6) as determining B.
In this paper we concentrate on the case $\mathcal{L}_\mu < \infty$ and $\mathcal{L}_\eta < \infty$ both sufficiently low, so that the described reversal in the determination directions of magnetic induction and velocity field takes place and boundary value problems are obtained for $\mathbf{B}$ as well as for $\mathbf{v}$.

The Complete Set of Equations. Applying the Helmholtz theorem for the decomposition of the velocity $\mathbf{v}$ into its irrotational and divergence-free parts and using flux representations for the divergence-free part of $\mathbf{v}$ and for $\mathbf{B}$ we may write

$$\mathbf{v} = \nabla S_M + \nabla \Psi_M \times \nabla \phi + C_M \nabla \phi \tag{7}$$

$$\mathbf{B} = \nabla \Psi \times \nabla \phi + C \nabla \phi \tag{8}$$

where $\phi = \varphi/(2\pi)$ in right-handed coordinates (R, φ, z). S_M is the velocity stream function S_M forced by the rate of expansion $\nabla \cdot \mathbf{v} \neq 0$. Let us consider the conservation laws for mass and energy and the equations (1) and (2) with the understanding that the representations (7) and (8) have been inserted. With κ being the thermal conductivity and $\mathbf{j} = \nabla \times \mathbf{B}/\mu_0$ the current density, conservation of mass and energy result in the Poisson equations (9) and (16) for S_M and T, where Q_M and E_M are given mass and energy sources. As to the remaining momentum balance and Ohm's law, we refer to the table

Projection	Equation of Motion	Ohm's Law
Δ	Δp	$\Delta \Phi$
$\nabla\phi \cdot$	$\Delta^* \Psi_M$	$\Delta^* \Psi$
$\nabla\phi \cdot \nabla \times$	$\Delta^* \Delta^* \Psi_M$	$\Delta^* C_M$

and apply the operations $\nabla^2 =: \Delta$, $\nabla\phi \cdot$ and $\nabla\phi \cdot \nabla\times$ to (1) and (2). Focussing attention upon the terms containing ∇p and $\nabla \Phi$, we see that the only operation not removing ∇p from (1) and $\nabla \Phi$ from (2) is Δ. The resulting equations (containing Δp and $\Delta \Phi$) appear below as the Poisson problems (14) and (15).
The second and third operations reflect the requirement that the toroidal and poloidal components of (1) and (2) must balance. Altogether, the following eight equations are obtained for $S_M, \Psi, C, C_M, \Psi_M, p, \Phi$ and T:

$$\Delta S_M = -\frac{1}{\rho} \nabla\rho \cdot \nabla S_M + \frac{1}{\rho} Q_M + \frac{1}{2\pi\rho R} \frac{\partial(\rho, \Psi_M)}{\partial(R, z)} \tag{9}$$

$$\eta \Delta^* \Psi = -\mu_0 \left\{ U + \frac{1}{2\pi R} \frac{\partial(\Psi, \Psi_M)}{\partial(R, z)} - \nabla S_M \cdot \nabla \Psi \right\} \tag{10}$$

$$\eta \Delta^* C = -\nabla\eta \cdot \nabla C - \frac{\mu_0 R}{2\pi} \left\{ \frac{\partial(C/R^2, \Psi_M)}{\partial(R, z)} - \frac{\partial(C_M/R^2, \Psi)}{\partial(R, z)} \right\} + \mu_0 C \Delta S_M \tag{11}$$

$$\mu\Delta^* C_M = -\frac{1}{2\pi\mu_0 R}\frac{\partial(\Psi, C)}{\partial(R, z)} + \rho\frac{\nabla S_M \cdot \nabla C_M}{R} \tag{12}$$

$$\mu\Delta^*\Delta^*\Psi_M = -\frac{R}{2\pi\mu_0}\left\{\frac{\partial((\Delta^*\Psi)/R^2, \Psi)}{\partial(R, z)} + \frac{\partial(C/R^2, C)}{\partial(R, z)}\right\} - \nabla\mu\cdot\nabla C_V \tag{13}$$

$$+C_V\left\{Q_M + \frac{1}{2\pi R}\frac{\partial(\rho, \Psi_M)}{\partial(R, z)}\right\} - 2\pi R\frac{\partial(\Delta S_M, \mu)}{\partial(R, z)} + \rho R^2\nabla S_M\cdot\nabla(C_V/R^2)$$

$$-\Delta p = \frac{1}{4\pi^2 R^2\mu_0}\left\{(\Delta^*\Psi)^2 + \nabla\Psi\cdot\nabla(\Delta^*\Psi) + C\Delta^* C + |\nabla C|^2\right\} + 2\nabla\cdot(\nabla\cdot(\mu\mathbf{D})) \tag{14}$$

$$\Delta\Phi = \frac{1}{4\pi R^2}(C_M\Delta^*\Psi - C\Delta^*\Psi_M + \nabla\Psi\cdot\nabla C_M - \nabla\Psi_M\cdot\nabla C) + \frac{1}{2\pi R}\frac{\partial(C, \eta/\mu_0 - S_M)}{\partial(R, z)} \tag{15}$$

$$\kappa\Delta T = -\nabla\kappa\cdot\nabla T + \frac{kT}{m(\gamma-1)}Q_M + \frac{k}{m(\gamma-1)}\rho\mathbf{v}\cdot\nabla T - \eta j^2 + p\Delta S_M - Q_E \tag{16}$$

where $\Delta^*\Psi \equiv R^2\nabla\cdot(R^{-2}\nabla\Psi)$ and $(\partial\alpha, \partial\beta)/\partial(R, z) \equiv 2\pi R(\nabla\beta\times\nabla\alpha)\cdot\nabla\phi$. The terms describing the nonlinear effects of plasma inertia have been omitted here for brevity.

Boundary Conditions. We use the notations $\partial F/\partial s = (\mathbf{t}\cdot\nabla F)|\partial D$ and $\partial F/\partial n = (\mathbf{n}\cdot\nabla F)\,|\partial D$ for any of the functions F under consideration, where D is some two-dimensional domain in a poloidal plane, $\mathbf{n}$ is the outward pointing normal on ∂D, s the poloidal arclength along ∂D and $\mathbf{t} = 2\pi R\nabla\phi\times\mathbf{n}$ the tangent vector in clockwise direction. In discussing (9) for given ρ and Q_M boundary conditions for S_M may be of Neumann or Dirichlet type. They are related to the normal and tangent components of the velocity on ∂D by

$$v_n = \partial S_M/\partial n + (1/(2\pi R))\partial\Psi_M/\partial s$$

$$v_t = \partial S_M/\partial s - (1/(2\pi R))\partial\Psi_M/\partial n$$

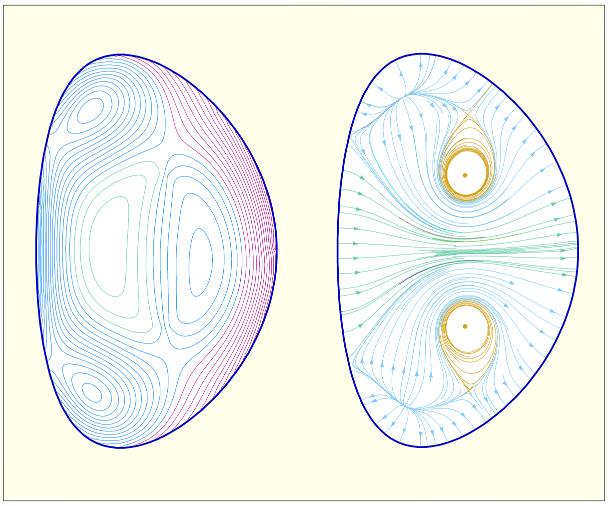

Figure 1: *Left:* Contour plot of the expansion rate S_M as solution of (9). *Right:* Field lines of poloidal flow with lower and upper stagnation points.

The fourth order operator appearing in (13) due to $\nabla\phi\cdot\nabla\times$ applied to (1) has the form $\Delta^*\Delta^*\Psi_M$. Splitting the latter into two of second order by introduction of the toroidal circulation of the vorticity C_V as an auxiliary variable

$$\nabla\times\mathbf{v} = \nabla C_M\times\nabla\phi + C_V\nabla\phi \qquad C_V = R^2\nabla\times\nabla\Psi_M\times\nabla\phi = -\Delta^*\Psi_M \tag{17}$$

This way the boundary conditions in connection with (13) are equivalent to such on C_V when solving (13), and others on Ψ_M when solving $\Delta^*\Psi_M + C_V = 0$. With Dirichlet conditions on both S_M and Ψ_M the normal derivatives of the resultant solutions can be calculated, and, using the expressions given above, also v_n and v_t. Conversely, when the latter are given, these expressions can be used for getting Dirichlet conditions on S_M and

Ψ_M in terms of v_n and v_t. Fig.1 illustrates a solution of (9), where for some mass density, mass source and vortex distribution $S_M|\partial D$ was determined such that the resulting flow satisfies the no-slip boundary condition $v_t = 0$. The boundary conditions on the remaining quantities C, C_M, C_V and Ψ are usually of Dirichlet type. Except for equation (10) which we consider as a free-boundary value problem for the poloidal magnetic flux Ψ describing a magnetically confined plasma in an external vacuum magnetic field, we assume all conditions to be imposed at a common control surface ∂D.

Solution of the Partial Differential Equations. For the efficient solution of the second order problems for both Neumann and Dirichlet boundary conditions a method was developed which generalizes an approach given in [3]. Let the problem to solve be $\Delta\Phi = \omega(x)$, $\Phi \,|\, \partial D = \Phi(s)$, where s is the arc length on ∂D, and ω and $\Phi(s)$ given. It is solved by considering two simpler problems with homogeneous boundary conditions on an auxiliary region $R \supset D$:

$$\Delta\Phi_{\square} = \omega(x),\ \Phi_{\square} \,|\, \partial R = 0, \qquad \Delta\Phi = \omega^*(x),\ \Phi \,|\, \partial R = 0 \tag{18}$$

where ω^* is a suitable extension of ω into R which, on the basis of the solution $\Phi_{\square}$, is constructed such that $\Phi \,|\, \partial D$ has the desired value. The Poisson problems with the Stokes operator Δ^* as well as Neumann boundary conditions have been handled in a similar manner.

The solution S_M presented in Fig.1 (left) was obtained by embedding D in a rectangular region R and then solving (18) using a cyclic reduction algorithm. Results obtained in this way have been compared with such from applying the multigrid methods of Mitchell [4] and Bank [5]. They have been found to be in good agreement for comparable region discretizations, where the embedding algorithm has turned out to be much faster.

Conclusions. Dissipative MHD equilibrium states with compressible fluid flow can be described by a group of partial differential equations, which for sufficiently low viscous and resistive Lundquist numbers requires the solution of certain number of elliptic Poisson problems. Unlike ideal equilibria there is a full set of prescribable physical boundary conditions and differently from steady MHD states with incompressible Navier-Stokes flow the vorticity-stream function problem of hydrodynamics does not exist. Depending on the competition between the source and the vortical flow terms in the mass conservation equation (9) complex poloidal flow patterns with source, saddle and stagnation points are observed (Fig.1), which separate confined from unconfined regions of the plasma.

[1] H. P. Zehrfeld, Resistive Equilibrium States of Axisymmetric Plasmas with Compressible Viscous Fluid Flow, in *26th EPS Conference on Controlled Fusion, Contributed Papers*, Maastricht, Netherlands, 1999.

[2] H. P. Zehrfeld and B. J. Green, Stationary Toroidal Equilibria at Finite Beta, Nuclear Fusion **12**(999), 569–575 (1972).

[3] B. L. Buzbee, F. W. Dorr, J. A. George, and G. H. Golub, Siam. J. Numer. Anal. **8**, 722 (1971).

[4] W. F. Mitchell, Adaptive refinement for arbitrary finite-element spaces with hierarchical bases, Journal of Computational and Applied Mathematics **36**, 65–78 (1991).

[5] R. E. Bank, *PLTMG: A Software Package for Solving Elliptic Partial Differential Equations – Users' Guide 8.0*, volume 8 of *Frontiers in Applied Mathematics*, SIAM, 1998.