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Kinetic theory and diffusion coefficients for plasma in a uniform magnetic field

I. Kourakis†, D. Carati

Université Libre de Bruxelles, Physique Statistique et Plasmas
Association Euratom - Etat Belge
C.P. 231, boulevard du Triomphe, B-1050 Brussels, Belgium
†email: ikouraki@ulb.ac.be

Abstract

Considering a test-particle weakly interacting with an electrostatic plasma in equilibrium *and* subject to a uniform magnetic field, a markovian Fokker-Planck-type kinetic equation is derived and explicit expressions for the diffusion and drift coefficients, depending on the magnetic field, are obtained. The explicit general form of the coefficients in the equation is presented and then explicitly calculated considering a Maxwellian reservoir distribution function and a Coulomb interaction law.

1. Introduction

In the context of plasma kinetic theory, we have undertaken a study of the dynamics of a charged particle interacting with a magnetized background plasma in equilibrium. Starting from first microscopic principles, a markovian Fokker-Planck-type kinetic equation (FPE) was derived in [1] and analytical expressions for the coefficients were obtained. The equation preserves the positivity of the phase-space distribution function, which was actually shown *not* to be the case in certain "markovianization" techniques proposed in the past. This new FP equation was thus suggested as a correct kinetic description of magnetized plasma and is now being used as a basis for the study of the influence of the magnetic field on the transport properties of plasma in various parameter regions and regimes - as compared, that is, to the standard Landau description for electrostatic plasma. A well-expected result was the explicit dependence of coefficients in the collision term on the reservoir equilibrium distribution function, the interaction potential and the external magnetic field. Relying on those results, we carry on here by explicitly evaluating the diffusion coefficients under the assumption that the reservoir state is Maxwellian and that the (long-range) interaction obeys a Coulomb law.

2. The model

We consider a test-particle (t.p.) Σ (charge $e_\Sigma = e$, mass $m_\Sigma = m$) surrounded by (and weakly coupled to) a homogeneous background plasma (N particles, of species $\alpha_j \in \{e, i, ...\}$, $\quad j = 1, 2, \ldots, N$) (the reservoir 'R'). The whole system is subject to a uniform external magnetic field. The equations of motion for the test-particle are:

$$\dot{\mathbf{x}} = \mathbf{v}\,; \qquad \dot{\mathbf{v}} = \frac{1}{m}\,[\frac{e}{c}(\mathbf{v}\times\mathbf{B}) \,+\, \lambda\,\mathbf{F_{int}}(\mathbf{x},\mathbf{v};\mathbf{X_R};t)] \tag{1}$$

where $\mathbf{X} = (\mathbf{x},\mathbf{v}) \equiv (\mathbf{x_\Sigma},\mathbf{v_\Sigma})$ and $\mathbf{X_R} \equiv \{\mathbf{X_j}\} = (\mathbf{x_j},\mathbf{v_j})$ denote the coordinates of the test- (Σ-) and reservoir ('R'-) particles respectively. The *interaction* force $\mathbf{F_{int}}(\mathbf{x},\mathbf{v};\mathbf{X_R};t)$

$= -\frac{\partial}{\partial \mathbf{x}} \sum V(|\mathbf{x} - \mathbf{x_j}|)$, actually the sum of interactions between $\Sigma-$ and $R-$ particles surrounding it, may be viewed as a random process, as the reservoir is assumed to be in equilibrium [2].

The zeroth-order (in λ) problem of motion yields the well-known (helicoidal) solution:

$$\mathbf{x}(t) = \mathbf{x}(0) + \mathbf{N}(t)\, \mathbf{v}(0) \qquad\qquad \mathbf{v}(t) = \mathbf{N}'(t)\, \mathbf{v}(0)$$

where

$$\mathbf{N}_i^{\alpha_i}(t) = \Omega^{-1} \begin{pmatrix} \sin \Omega t & s\,(1 - \cos \Omega t) & 0 \\ s\,(\cos \Omega t - 1) & \sin \Omega t & 0 \\ 0 & 0 & \Omega t \end{pmatrix} \tag{2}$$

$\Omega = \Omega^{\alpha_j} \equiv \frac{|e_{\alpha_j}| B}{m_{\alpha_j} c}$ is the gyro-frequency of particle j and $s = s_{\alpha_j} = \frac{e_{\alpha_j}}{|e_{\alpha_j}|} = \pm 1$ is the *sign* of e_j (the t.p. is understood where the subscript is omitted in the following).

3. Statistical formulation - a kinetic equation

The test-particle's reduced distribution function is $f(\mathbf{x}, \mathbf{v}; t) = (I,\, \rho)_R \equiv \int_{\Gamma_R} d\mathbf{X_R}\, \rho$, where $\rho = \rho(\{\mathbf{X}, \mathbf{X_R}\}; t)$ ($F = F(\mathbf{X_R})$) denotes the total (reservoir) phase-space distribution function (d.f.), which is normalized to unity: $\int d\mathbf{X}\, \rho = 1$ ($\int d\mathbf{X_R}\, F = 1$). By assuming the interactions to be weak, the BBGKY hierarchy of equations is truncated to 2nd order in λ; by neglecting initial correlations, f is found to obey a Non-Markovian Master Equation. Following an approach developed in the past in the context of open non-equilibrium statistical mechanical systems [3], the latter was shown in [1] to lead to a Fokker-Planck-type equation of the form:

$$\begin{aligned} \frac{\partial f}{\partial t} + \mathbf{v} \frac{\partial f}{\partial \mathbf{x}} + \frac{e}{mc} (\mathbf{v} \times \mathbf{B}) \frac{\partial f}{\partial \mathbf{v}} = {} & \Big[\Big(\frac{\partial^2}{\partial v_x^2} + \frac{\partial^2}{\partial v_y^2} \Big) \Big[D_\perp(\mathbf{v}) f \Big] + \frac{\partial^2}{\partial v_z^2} [D_\parallel(\mathbf{v}) f] \\ & + 2\, s\, \Omega^{-1} \Big[\frac{\partial^2}{\partial v_x \partial y} - \frac{\partial^2}{\partial v_y \partial x} \Big] \Big[D_\perp(\mathbf{v}) f \Big] + \Omega^{-2}\, [D_\perp^{(XX)}(\mathbf{v})] (\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}) f \\ & - \frac{\partial}{\partial v_x} \Big[\mathcal{F}_x(\mathbf{v})\, f \Big] - \frac{\partial}{\partial v_y} \Big[\mathcal{F}_y(\mathbf{v})\, f \Big] - \frac{\partial}{\partial v_z} \Big[\mathcal{F}_z(\mathbf{v})\, f \Big] \\ & + s\, \Omega^{-1}\, \mathcal{F}_y(\mathbf{v}) \frac{\partial}{\partial x} f - s\, \Omega^{-1}\, \mathcal{F}_x(\mathbf{v}) \frac{\partial}{\partial y} f \end{aligned} \tag{3}$$

[4]. Note that all coefficients are functions of $\mathbf{v}$ (actually of $\{v_\perp, v_\parallel\}$) only. Therefore, by integrating over space $\{\mathbf{x}\}$ one readily obtains the equation:

$$\begin{aligned} \frac{\partial f}{\partial t} + \frac{e}{mc} (\mathbf{v} \times \mathbf{B}) \frac{\partial f}{\partial \mathbf{v}} = {} & \Big[\Big(\frac{\partial^2}{\partial v_x^2} + \frac{\partial^2}{\partial v_y^2} \Big) \Big[D_\perp(\mathbf{v}) f \Big] + \frac{\partial^2}{\partial v_z^2} [D_\parallel(\mathbf{v}) f] \\ & - \frac{\partial}{\partial v_x} \Big[\mathcal{F}_x(\mathbf{v})\, f \Big] - \frac{\partial}{\partial v_y} \Big[\mathcal{F}_y(\mathbf{v})\, f \Big] - \frac{\partial}{\partial v_z} \Big[\mathcal{F}_z(\mathbf{v})\, f \Big] \end{aligned} \tag{4}$$

(an equation of a similar form has appeared in earlier works [5]).

4. Coefficients

The coefficients in (3) are defined by:

$$\left\{ \begin{array}{c} D_\perp \\ D_\angle \\ D_\perp^{(XX)} \\ D_\| \end{array} \right\} = \sum_{\alpha'} \frac{n_{\alpha'}}{m^2} (2\pi)^3 \int_0^t d\tau \int d\mathbf{v_1}\, \phi_{eq}^{\alpha'}(\mathbf{v_1}) \int d\mathbf{k}\, \tilde{V}_k^2\, e^{ik_n N_{nm}^{\alpha}(\tau) v_m}\, e^{-ik_n N_{nm}^{\alpha'}(\tau) v_{1,m}} \left\{ \begin{array}{c} \frac{1}{2}\, k_\perp^2\, \cos \Omega^\alpha \tau \\ (-s^\alpha)\, \frac{1}{2}\, k_\perp^2\, \sin \Omega^\alpha \tau \\ k_\perp^2 (1 + \frac{1}{2} \cos \Omega^\alpha \tau) \\ k_\|^2 \end{array} \right\} \qquad (5)$$

(a summation over n, m is understood) and

$$\mathcal{F}_x = (1+\mu)\, (\frac{\partial D_\perp}{\partial v_x} + \frac{\partial D_\angle}{\partial v_y}) \qquad \mathcal{F}_y = (1+\mu)\, (-\frac{\partial D_\angle}{\partial v_x} + \frac{\partial D_\perp}{\partial v_y})$$
$$\mathcal{F}_z = (1+\mu)\, \frac{\partial D_\|}{\partial v_z} \qquad (6)$$

where v_i ($v_{1,i}$), $i = 1, 2, 3$ denote the velocity coordinates of the test-particle (reservoir-particle) of species $\alpha_\Sigma \equiv \alpha$ ($\alpha_1 = \alpha'$) respectively and $\tilde{V}_k$ stands for the Fourier transform of the interaction potential (remember that $V = V(|\mathbf{r}|) = V(r)$ implies that $V = \tilde{V}(|\mathbf{k}|) \equiv \tilde{V}_k$); obviously $k_\perp^2 = k_x^2 + k_y^2$, $k_\| = k_z$. Note the explicit dependence on the magnetic field through $\Omega = \Omega_i^{\alpha_i}$ ($i = \Sigma^\alpha, 1^{\alpha'}$) and also on the form of the reservoir equilibrium d.f. $\phi_{eq} = \phi_{eq}(v_\perp, v_\|)$ and the interaction potential $V(r)$.

The v_1- integration in (5) can be carried out at this stage, once one assumes an analytic form for the equilibrium reservoir distribution function (d.f.) ϕ_{eq}. Here, it will be explicitly taken to be a Maxwellian of the form:

$$\phi_{Max}^{\alpha'}(v_1) = \prod_{i=1,2,3} \phi_0^{(i,\alpha')}\, e^{-v_{1,i}^2/\sigma_i^{\alpha'}} \qquad (7)$$

($\phi_0^{(i)} = (\frac{m_{\alpha'}}{2\pi T_{\alpha'}^{(i)}})^{1/2} \equiv \frac{1}{\sqrt{\pi \sigma_i^{\alpha'}}}$; $\sigma_i^{\alpha'} \equiv 2\, {v_{i,th}^{\alpha'}}^2 \equiv \frac{2T_i^{\alpha'}}{m_{\alpha'}}$ $\forall i \in \{1, 2, 3\} \equiv \{x, y, z\}$; we assume here that $\sigma_1^{\alpha'} = \sigma_2^{\alpha'} = \sigma_\perp$, $\sigma_3^{\alpha'} = \sigma_\|$; the summation over particle species α' is omitted in the following). The calculation yields:

$$\left\{ \begin{array}{c} D_\perp \\ D_\angle \\ D_\perp^{(XX)} \end{array} \right\} = \frac{n_\alpha}{m^2} (2\pi)^4\, e^{-v_m^2/\sigma_m} \int_0^t d\tau \left(\int d\mathbf{k}\, e^{-\sigma_m\, q_m^2/4}\, k_\perp^2\, \tilde{V}_k^2 \right) \left\{ \begin{array}{c} \frac{1}{2} \cos \Omega\tau \\ (-s)\, \frac{1}{2}\, \sin \Omega\tau \\ 1 + \frac{1}{2}\, \cos \Omega\tau \end{array} \right\}$$
$$D_\| = \frac{n_\alpha}{m^2} (2\pi)^4\, e^{-v_m^2/\sigma_m} \int_0^t d\tau \left(\int d\mathbf{k}\, e^{-\sigma_m\, q_m^2/4}\, k_\|^2\, \tilde{V}_k^2 \right) \qquad (8)$$

where

$$\sigma \equiv (\sigma_1, \sigma_2, \sigma_3) \qquad q_m(\mathbf{k}; \tau; v) = \sum_{n=1}^{3} k_n N_{nm}(\tau) - i\frac{2 v_m}{\sigma_m} \qquad (9)$$

Note that the integral within parenthesis possesses a cylindrical symmetry due to the existence of the $\mathbf{N}$ matrix in it (cf. (2)); in the absence of an external magnetic (or *any*) field, it reduces to a *spherically* symmetric form, as $\mathbf{N}(\tau) \to \tau \mathbf{I}$.

As a matter of fact, relation (8) holds as it stands for *any* particular form of $V(r)$. Let us now explicitly assume that the (long-range) interaction potential $V(r)$ is a Coulomb-type long-range potential: $V(r) = \frac{V_0}{r}$ i.e. $\tilde{V}_k = \frac{\tilde{V}_0}{k_\perp^2 + k_\|^2}$ (V_0, $\tilde{V}_0$ are appropriate constant quantities). The coefficients in (5) (actually functions of $\{v_\perp, v_\|, t;\, \sigma_\perp, \sigma_\|, \Omega\}$) now become:

$$\left\{ \begin{array}{c} \left\{ \begin{array}{c} D_\perp \\ D_\angle \\ D_\perp^{(XX)} \end{array} \right\} \\ D_\| \end{array} \right\} = \frac{n_\alpha}{m^2} (2\pi)^4\, \tilde{V}_0^2 \int_0^t d\tau \int_0^\infty dk_\perp\, e^{-\sigma_\perp \frac{k_\perp^2}{\Omega^2} \sin^2 \frac{\Omega\tau}{2}}\, J_O(2 \frac{k_\perp v_\perp}{\Omega} \sin \frac{\Omega \tau}{2})$$

$$\left\{ \begin{array}{c} \left\{ F_\perp \right\} \\ F_\| \end{array} \right\} \left\{ \begin{array}{c} \left\{ \begin{array}{c} \frac{1}{2} \cos \Omega\tau \\ \frac{-s}{2} \cos \Omega\tau \\ 1 + \frac{1}{2} \cos \Omega\tau \end{array} \right\} \\ 1 \end{array} \right\} \quad (10)$$

where the functions $F = F_{\{\perp,\|\}}(k_\perp, v_\|, \tau;\, \sigma_\|)$ are given by:

$$F_{\{\perp,\|\}} = \pm \frac{\sqrt{\pi}}{2} \sqrt{\sigma_\|}\, k_\perp\, \tau\, e^{-v_\|^2/\sigma}$$
$$+ \frac{\pi}{4}\, e^{\sigma_\| k_\perp^2 \tau^2/4} \sum_{s=+1,-1} \left[e^{s k_\perp v_\| \tau} \left(1 \mp \sigma_\| k_\perp^2 \tau^2/2 \mp s k_\perp v_\| \tau\right) Erfc(\frac{1}{2} \sqrt{\sigma_\|} k_\perp \tau + s \frac{v_\|}{\sqrt{\sigma}}) \right] \quad (11)$$

the upper (lower) signs corresponding to the $\perp -$ ($\| -$) parts respectively. $Erfc(x)$ denotes the *complementary* error function: $Erfc(x) = 1 - Erf(x) \equiv 1 - \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$.

Note that the integrand vanishes at infinity i.e. at $k_\perp \to \infty$ (and also at $\tau \to \infty$). Futhermore, the limit of the integrands at $k_\perp \to 0$ is finite (and the same holds for $\tau \to 0$).

5. Conclusions

In conclusion, we have reported eq.(3) as a correct kinetic description, from first principles, of the dynamics of magnetized plasma, at least up to second order in the (weak) interaction. In the homogeneous case, the previous result is obtained; furthermore, in the absence of external field, the Landau equation is recovered. Once the role of the magnetic field on transport properties is elucidated, realistic generalizations, taking into account field-inhomogeneities, should be taken into account; work in this direction is in progress.

References

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