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Statistical Theory of Subcritically-Excited Strong Turbulence in Inhomogeneous Plasmas

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1. Introduction

The study of strong turbulence in high temperature plasmas is an important issue of physics: the problems of statistical physics for systems far from thermodynamic equilibrium remain quite open, in contrast to those near thermodynamic equilibrium in which the principles that govern fluctuations (i.e., equipartition of energy, Einstein relation, fluctuation-dissipation (FD) theorem, etc.) are established.[1)]

Recently, a statistical description and the extended analyses have been developed for a self-sustained strong turbulence which is caused by the subcritically excited interchange mode.[2)] A Langevin equation for a dressed test mode, in which the nonlinear interactions are divided into the drag term (coherent interactions) and the random noise term (incoherent ones), is formulated. Imposing ansatz (1) of a large numbers of degrees of freedom in the turbulence (extensiveness) and (2) of the randomness of self-noise, the turbulent level and decorrelation rate of turbulence and the auto- and cross-correlation functions have been solved. The extended FD-theorem (Einstein relation) has been explicitly described for a plasma turbulence by the nonequilibrium-parameter (the gradient) of the system. We also extended the analysis, including the effects of thermal fluctuations.[2)] Their coherent interactions with the plasma collective mode (e.g., CDIM)[3)] are represented by the collisional drags, and their incoherent interactions are considered to be a random noise of plasma temperature T.

In this paper formulation is presented by deriving an Fokker-Planck equation for the probability distribution function.[4)] Equilibrium distribution function of fluctuations is obtained. Transition from the thermal fluctuations, that is governed by the Boltzmann distribution, to the turbulent fluctuation is clarified. The distribution function for the turbulent fluctuation has tail component and the width of which is in the same order as the mean fluctuation level itself. The Lyapunov function is constructed for the strongly turbulent plasma, and it is shown that an approach to a certain equilibrium distribution is assured. From the Fokker-Planck equation the transition probability between the thermal fluctuation and turbulent fluctuation is derived. The formula recovers the Arrhenius law in the thermodynamical equilibrium limit. The power law dependence of the transition probability is obtained on the distance between the pressure gradient and the critical gradient for linear instability. Thus a new type of critical exponent is explicitly deduced in the phenomena of subcritical excitation of turbulence.

2. Basic Equation and Statistical Approach

2.1 Plasma model and basic equation

The dynamics of micro fluctuations are studied in the presence of the global inhomogeneity of the plasma pressure. Quantities that are averaged over the (y, z) -plane are denoted by the suffix 0, as p_0 and ϕ_0. $\phi = \phi_0 + \tilde{\phi}$, $J = J_0 + \tilde{J}$ and $p = p_0 + \tilde{p}$. The pressure and electrostatic potential could be inhomogeneous (i.e., inhomogeneous in the $\hat{x}$-direction) in the global scale. Parameters ∇p_0 and $\nabla_\perp^2 \phi_0$ together with Ω' represent the inhomogeneity of the system. The scale separation is introduced as $\left| p_0^{-1}\, \partial p_0/\partial t \right| << \left| \tilde{p}^{-1}\, \partial \tilde{p}/\partial t \right|$ and $\left| p_0^{-1} \nabla p_0 \right| << \left| \tilde{p}^{-1} \nabla \tilde{p} \right|$.

We consider the thermal fluctuations in the range of ω_p and the time scales between microscopic mode (CDIM) are well separated. In the thermal fluctuations, coherent parts to the microscopic CDIM are given by the collisional transport coefficients μ_c, μ_{ec} and χ_c (the ion viscosity, electron viscosity and thermal diffusivity, respectively). Incoherent parts are considered to be a random noise and expressed as $\tilde{S}_{th}$.[1] The relation between them is described by fluctuation dissipation theorem.[1]

2.2 Langevin equation for turbulent fluctuations

A Langevin equation is deduced by use of the renormalized eddy viscosity and a random coupling model (RCM).[5] Basic equation is given by

$$\frac{\partial}{\partial t}\boldsymbol{f} + \mathcal{L}^{(0)}\boldsymbol{f} = \mathcal{N}(\boldsymbol{f}) + \tilde{\boldsymbol{S}}_{th}, \tag{1}$$

$$\mathcal{L}^{(0)} = \begin{pmatrix} -\mu_c\nabla_\perp^2 & -\nabla_\perp^{-2}\nabla_{||} & -\nabla_\perp^{-2}\Omega'\frac{\partial}{\partial y} \\ \xi\nabla_{||} & -\mu_{ec}\nabla_\perp^2 & 0 \\ -\frac{dp_0}{dx}\frac{\partial}{\partial y} & 0 & -\chi_c\nabla_\perp^2 \end{pmatrix}, \quad \boldsymbol{f} = \begin{pmatrix} \phi \\ J \\ p \end{pmatrix}.$$

A projection operator $\mathcal{P}$ is introduced to divide the nonlinear interactions into the drag and others.[2] Equation (1) is written as

$$\frac{\partial}{\partial t}\boldsymbol{f} + \mathcal{L}_0\boldsymbol{f} - \mathcal{P}\mathcal{N}(\boldsymbol{f}) = (\boldsymbol{1} - \mathcal{P})\mathcal{N}(\boldsymbol{f}) + \tilde{\boldsymbol{S}}_{th} \tag{2}$$

($\boldsymbol{1}$ is a unit operator) where the nonlinear drag is written in an apparent linear term as

$$\mathcal{P}\mathcal{N}(\boldsymbol{f}) = \begin{pmatrix} \mu_N\nabla_\perp^2 f_1 \\ \mu_{Ne}\nabla_\perp^2 f_2 \\ \chi_N\nabla_\perp^2 f_3 \end{pmatrix} = -\begin{pmatrix} \gamma_1 f_1 \\ \gamma_2 f_2 \\ \gamma_3 f_3 \end{pmatrix} \tag{3}$$

and the rest part is rewritten as $\tilde{\boldsymbol{S}} = (\boldsymbol{1} - \mathcal{P})\mathcal{N}(\boldsymbol{f})$. A Langevin equation is derived as[2,4]

$$\frac{\partial}{\partial t}\boldsymbol{f} + \mathcal{L}\boldsymbol{f} = \tilde{\boldsymbol{S}} + \tilde{\boldsymbol{S}}_{th} \text{ with } \mathcal{L}_{ij} = \mathcal{L}_{ij}^{(0)} + \gamma_i\delta_{ij}, \tag{4}$$

$$\gamma_{i,\,k} = -\sum_{\Delta} M_{i,\,kpq}M^*_{i,\,qkp}\theta^*_{qkp}\left|\tilde{f}^2_{1,p}\right|. \tag{5}$$

The self-noise has a much shorter correlation time (due to RCM) and is approximated to be given by the Gaussian white noise term $w(t)$ as

$$\tilde{S}_{i,\,k} = w(t)\sum_{\Delta} M_{i,\,kpq}\sqrt{\theta_{kpq}}\,\zeta_{1,\,p}\zeta_{i,\,q}\,. \tag{6}$$

2.3 Solution of Langevin equation and Statistical average

To solve the Langevin eq.(4), an ansatz of large number of degrees of freedom in random modes is introduced. The general solution is formally given as

$$f(t) = \exp\left[-\mathcal{L}t\right] f(0) + \int_0^t \exp\left[-\mathcal{L}(t-\tau)\right]\left\{\tilde{S}(\tau) + \tilde{S}_{th}(\tau)\right\} \mathrm{d}\tau \qquad (7)$$

The eigenvalue of CDIM is determined by: $\det\left(\lambda \mathbf{1} + \mathcal{L}\right) = \left(\lambda+\lambda_1\right)\left(\lambda+\lambda_2\right)\left(\lambda+\lambda_3\right) = 0$.

The matrix $\exp\left[-\mathcal{L}(t-\tau)\right]$ in eq.(7) is explicitly expressed as[2)]

$$\exp\left[-\mathcal{L}\left(t-\tau\right)\right] = \boldsymbol{A}^{(1)} \exp\left(-\lambda_1\left(t-\tau\right)\right) + \boldsymbol{A}^{(2)} \exp\left(-\lambda_2\left(t-\tau\right)\right) + \boldsymbol{A}^{(3)} \exp\left(-\lambda_3\left(t-\tau\right)\right) \quad . \qquad (8)$$

where $\boldsymbol{A}^{(i)}$ is a projection operator (rank=1) to extract the eigenmode with λ_i which statisfies $\boldsymbol{A}^{(i)} \; \boldsymbol{A}^{(i)} = \boldsymbol{A}^{(i)}$ and $\boldsymbol{A}^{(i)} \; \boldsymbol{A}^{(j)} = 0$ for $i \neq j$.

Long-time-averaged values are calculated, where the initial condition in eq.(7) is unimportant and is neglected. We write

$$f_i(t) f_j(t) = \int_0^t \mathrm{d}\tau \int_0^t \mathrm{d}\tau' \left\{\exp[-\mathcal{L}(t-\tau)]\left\{\tilde{S}(\tau) + \tilde{S}_{th}(\tau)\right\}\right\}_i \left\{\exp[-\mathcal{L}(t-\tau')]\left\{\tilde{S}(\tau') + \tilde{S}_{th}(\tau')\right\}\right\}_j^*$$

where the relation eq.(6) for $\tilde{S}$ and eq.(8) should be substituted and we have the extended FD theorem of the second kind as

$$\left\langle f_i f_j \right\rangle = \frac{1}{2\lambda_1} \sum_{i', j'} A_{ii'} \left\{\left\langle \tilde{S}_{i'} \tilde{S}_{j'} \right\rangle + \left\langle \tilde{S}_{th,\, i'} \tilde{S}_{th,\, j'} \right\rangle\right\} A_{jj'}^{*}. \qquad (9)$$

/3. Fokker-Planck equation

3.1 Fokker-Planck equation and Probability distribution function

For the most unstable branch with the eigenvalue $-\lambda_1$, we write the Fokker-Planck equation (FP eq.) for the probability distribution function (PDF), assuming the noise is gaussian white as

$$\frac{\partial}{\partial t} P = \sum_k \frac{\partial}{\partial \phi_k} \left(\lambda_{1,k} \phi_k + \frac{1}{2} \hat{g}_k \frac{\partial}{\partial \phi_k} \hat{g}_k\right) P \qquad (10)$$

where $\tilde{S}_{j,k} = w(t)\left(g_{j,k} + g_{th,j,k}\right)$ and the transformation $g_k = \Re e\left(\sum_{j=1}^{3} A_{1j} g_{j,k}\right)$ with $\hat{g}_k^2 = g_k^2 + g_{th,k}^2$ is used. The equilibrium distribution function is solved by setting $\partial / \partial t = 0$ as

$$P_{eq}\left(\left\{\phi_k\right\}\right) = \bar{P} \prod_k \frac{1}{\hat{g}_k} \exp\left\{-\int^{\phi_k} 2\lambda_{1,k} \phi_k \, \hat{g}_k^{-2} \, \mathrm{d}\phi_k\right\} \qquad (11)$$

This formula reduces to the case with minimum entropy production rate if we neglect the variance (noise) of turbulence $g_k^2 \to 0$. Further neglection of turbulent drag term (eg.(3)) in eq. (11) gives the Boltzmann distribution with thermal fluctuations as

$$P_{eq}\left(\left\{\mathcal{E}_k\right\}\right) \propto \frac{1}{\sqrt{\hat{T}}} \exp\left\{-\sum_k \frac{\mathcal{E}_k}{\hat{T}}\right\} \qquad (12)$$

where $\mathcal{E}_k$ is the kinetic energy of fluetuations. Note that the Lyapunov function is constructed from eg. (10) and the H-theorem is guaranteed.

3.2 Energy distribution function and Transition probability in Turbulencs

We adopt the turbulent energy as the coarse grained variable defined by $\mathcal{E} \equiv \frac{1}{2} \sum_k k_\perp^2 \phi_k^2$ and obtain the reduced FP eq. as

$$\frac{\partial}{\partial t} P(\mathcal{E}) = \frac{\partial}{\partial \mathcal{E}} \left(2\Lambda \mathcal{E} + \frac{1}{2} g \frac{\partial}{\partial \mathcal{E}} g \right) P(\mathcal{E}) \tag{13}$$

where $\Lambda \equiv \sum_k \lambda_{1,k} k_\perp^2 \phi_k^2 / \sum_k k_\perp^2 \phi_k^2$ is the averaged decorrelation (damping) rate and the combined noise term is given by $g^2 = \sum_k 2\mu_{vc} \hat{T} k_\perp^4 \phi_k^2 + \sum_k \left(\sum_{j=1}^{3} A_{1j} g_{j,k} \right)^2 k_\perp^4 \phi_k^2$.

The equilibrium distribution function is given by

$$P_{eq}(\mathcal{E}) = \bar{P} \frac{1}{g} \exp\left(- S(\mathcal{E}) \right) \tag{14}$$

where $S(\mathcal{E}) = \int_0^{\mathcal{E}} 4\Lambda g^{-2} \mathcal{E} \mathrm{d}\mathcal{E}$ plays an effective potential with respect to $\mathcal{E}$. At the turbulent energy level which satisfies the extremum of $S(\mathcal{E})$, quasi-steady state exists.

In the nonlinear dissipative interchange mode turbulence, the subcritical excitation is possible. The typical effective potential and the distribution function are given in ref 4). Furthermore, if the potential has two minima (say $\mathcal{E} = \mathcal{E}_{\mathcal{A}}$, and $\mathcal{E} = \mathcal{E}_{\mathcal{B}}$), the transition between the two quasi-states is possible. The transition probability from A to B is calculated to be

$$r_{A \to B} = \frac{\sqrt{\lambda_0}}{2\sqrt{\pi}} \frac{1}{\Delta \mathcal{E}_A} g(\mathcal{E}_A) \exp\left\{ S(\mathcal{E}_A) - S(\mathcal{E}_C) \right\} \tag{15}$$

where the potential around the maximum point $\mathcal{E}_C$ ($\mathcal{E}_A < \mathcal{E}_C < \mathcal{E}_B$) is approximated as $S(\mathcal{E}) \simeq S(\mathcal{E}_c) - \lambda_0 g^{-2}(\mathcal{E}_c) \left(\mathcal{E}_c - \mathcal{E} \right)^2$. This (eq. (15)) is an extension of Arrhenius law for the thermal fluctuation to the one for turbulent fluctuations.

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