CONTRIBUTION TO THE STUDY OF THE NON-GAUSSIAN DYNAMICS OF STOCHASTIC MAGNETIC FIELD LINES IN A TOROIDAL GEOMETRY (TOKAMAK)

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Abstract

In this work, we study the non-Gaussian dynamics of the magnetic field lines of a plasma confined in a toroidal configuration (tokamak). We have used the Fourier paths technique in order to calculate the kurtosis of the magnetic field lines. We have also calculate numerically this quantity. The magnetic field lines are described by the standard mapping. A comparison between our analytical and numerical calculations have been done.

1. Introduction

In the thermonuclear fusion, the quality of energy and particle confinement in a tokamak is strongly dependent on the structure of the magnetic field lines. The study of stochastic magnetic field lines diffusion has been done in several works (i.e. [1-3]). These studies are based on the assumption that the behavior of magnetic field lines is Gaussian. However, a numerical study [4], based on the calculation of the quantity called kurtosis, has shown the non-Gaussian dynamics of stochastic magnetic field lines in the interval [0,70] of the stochasticity parameter and for values larger than 70, the dynamics becomes Gaussian. A similar result was found by Zimbardo and al. [5].

In this work ,we calculate analytically the kurtosis using the technique of Fourier paths. This technique has been introduced in order to calculate the velocity moments and have only been applied to systems for which the standard mapping is a global representation. It has been used by Rechester and al [6] to calculate the diffusion coefficient of particles in the presence of external stochasticity and by Abarbanel [7] without introduction of external stochasticity. Our calculation is based on the method developed by Abarbanel given in the reference [8].

In this work,we contribute to study the influence of the stochasticity, without taking account to the external stochasticity, on the behavior of magnetic field lines. These field lines will be described by the standard mapping.

In the section 2, we introduce the definition of the kurtosis for the stochastic magnetic field lines and we develop the analytical calculation of the kurtosis. In the section 3, we discuss the results of our calculation.

2. Analytical calculation

To describe the magnetic field lines behavior, it can be used the Poincaré representation that is reduced to discrete mapping [3]. In this work, we have used the standard mapping given by

$$I_{k+1} = I_k + K\sin(\theta_k),$$
$$\theta_{k+1} = \theta_k + I_{k+1}, \tag{1}$$

with $I_k = 2\pi\psi_k$ and K is the parameter of stochasticity.

We have considered the radial displacement $\Delta I = I - I_0$ as a random variable. The parameter kurtosis writes therefore as

$$kurt = \frac{\langle(\Delta I)^4\rangle}{\langle(\Delta I)^2\rangle^2}. \tag{2}$$

For a non-Gaussian dynamics this quantity is different to 3. In the opposite case, i.e. $kurt = 3$, we have a Gaussian behavior.

To obtain a analytical expression of the kurtosis, we calculate $M_4 = \langle(\Delta I)^4\rangle$ and $M_2 = \langle(\Delta I)^2\rangle$ in terms of the conditional probability w that an initial state (I_0, θ_0) at n=0 evolves to a final state (I, θ) at step n [8]

$$M_i = \int w(I,\theta,n/I_0,\theta_0,0)(I - I_0)^i \, dI\, d\theta\,. \tag{3}$$

w satisfy the following recursion property

$$w(I,\theta,n/I_0,\ \theta_0,0) = \int dI' d\theta' w(I,\theta,n/I',\theta',n-1) w(I',\theta',n-1/I_0,\theta_0,0)\,, \tag{4}$$

where, from the standard mapping

$$w(I,\theta,n/I',\theta',n-1) = \delta(I - I' - K\sin\theta')\delta(\theta - \theta' - I' - K\sin\theta')\,. \tag{5}$$

Expanding w in a Fourier series in θ and a Fourier integral in I,

$$w\ \ (I,\theta,n/I_0,\theta_0,0) = \sum_m \int dq \exp(im\theta + iIq) a_n(m,q)\,. \tag{6}$$

M_4 and M_2 can be written as

$$M_4 = E_4 - 4I_0 E_3 + 6{I_0}^2 E_2 - 4{I_0}^3 E_1 + {I_0}^4 E_0\,,$$
$$M_2 = E_2 - 2I_0 E_1 + {I_0}^2 E_0\,, \tag{7}$$

with $E_i = \int W(I,\theta,n/I_0,\theta_0,0) I^{\ i} \, dI\, d\theta$.

Using equation (6) in the expression of E_i, we obtain

$$E_4 = (2\pi)^2 \frac{\partial^4}{\partial q^4} a_n(0,q)\Big|_{q=0}\,,$$

$$E_3 = i(2\pi)^2 \frac{\partial^3}{\partial q^3} a_n(0,q)\Big|_{q=0}\,,$$

$$E_2 = -(2\pi)^2 \frac{\partial^2}{\partial q^2} a_n(0,q)\Big|_{q=0}\,,$$

$$E_1 = -i(2\pi)^2 \frac{\partial}{\partial q} a_n(0,q)\Big|_{q=0}\,,$$

$$E_0 = (2\pi)^2 a_n(0,q)\big|_{q=0}\,. \tag{8}$$

In order to explicit M_4 and M_2, we need to calculate the Fourier coefficient a_n. They satisfy the following recursion relation

$$a_n(m_n,q_n) = \sum_{l_n} J_{l_n}(K|q_{n-1}|)a_{n-1}(m_{n-1},q_{n-1}), \tag{9}$$

with

$$m_n = m_{n-1} - l_n sgnq_{n-1},$$
$$q_n = q_{n-1} - m_n. \tag{10}$$

The sgn function is defined as

$$sgnq_r = \begin{cases} +1 & if \quad q_r \geq 0 \\ -1 & if \quad q_r < 0 \end{cases}.$$

Iterating *n* times the equation (9) yields a_n in terms of a_0

$$a_n(m_n,q_n) = \sum_{l_n......l_1} J_{l_n}(K|q_{n-1}|)..J_{l_1}(K|q_0|)a_0(m_0,q_0), \tag{11}$$

with

$$a_0(m,q) = \frac{1}{(2\pi)^2}\exp(-im\theta_0 - iqI_0). \tag{12}$$

The set of *n* integers $\{l_n.......l_1\}$defines through (10) a path in the (*m*,*q*) Fourier space.
From the relations (8), the path must end at $m_n = 0$ and $q_n = 0$.
We have shown using the Fourier paths technique that for large values of *n*, $M_4$and M_2 take the following expressions

$$\begin{aligned} M_4 = {} & 3n^2K^4[\frac{1}{4} - J_2(K) - J_1^{\,2}(K) + J_2^{\,2}(K) + J_3^{\,2}(K)] \\ & + 3nK^4[-\frac{1}{8} + 2J_2(K) + 3J_1^{\,2}(K) - 2J_2^{\,2}(K) - 3J_3^{\,2}(K) \\ & + J_4(2K) + J_2^{\,2}(2K) + J_4^{\,2}(2K) + J_6^{\,2}(2K) \\ & - 2J_2(K)J_4(2K) - 2J_1(K)J_5(3K) - 2J_5(K)J_7(3K)], \end{aligned}$$

$$M_2 = nK^2(\frac{1}{2} - J_2(K)). \tag{13}$$

Finally the kurtosis is

$$\begin{aligned} kurt = {} & \frac{M_4}{M_2^{\,2}} \\ = {} & \{ 3n^2K^4[\frac{1}{4} - J_2(K) - J_1^{\,2}(K) + J_2^{\,2}(K) + J_3^{\,2}(K)] \\ & + 3nK^4[-\frac{1}{8} + 2J_2(K) + 3J_1^{\,2}(K) - 2J_2^{\,2}(K) - 3J_3^{\,2}(K) \\ & + J_4(2K) + J_2^{\,2}(2K) + J_4^{\,2}(2K) + J_6^{\,2}(2K) \\ & - 2J_2(K)J_4(2K) - 2J_1(K)J_5(3K) - 2J_5(K)J_7(3K)] \} \\ & \times 1 \big/ n^2K^4(\frac{1}{2} - J_2(K))^2. \end{aligned} \tag{14}$$

We have represented this result in the figure 1, where we have also represented a numerical calculation of the kurtosis. To calculate numerically the kurtosis we have used the following relations

$$M_i = \frac{1}{N}\sum_{i=1}^{N}(I_n^i - I_0^i)^i \qquad (15)$$

where N=3000 is the magnetic field lines and n=50 is the number of iterations.

3. Conclusion

In this work, we have studied the stochasticity influence on the non-Gaussian behavior of the magnetic field lines in a the tokamak. We have calculate analytically and numerically the kurtosis of the magnetic field lines. These calculations have shown they have a Gaussian behavior only for values of the stochasticity parameter larger than to 60. For $K < 60$, the magnetic field cannot be described by Gaussian randum function.

We note finally that we have taken an interest to the case of stochasticity parameter K great than 4. For $K < 4$, the effects of KAM surfaces became increasingly important, requiring the calculation of more Fourier paths.

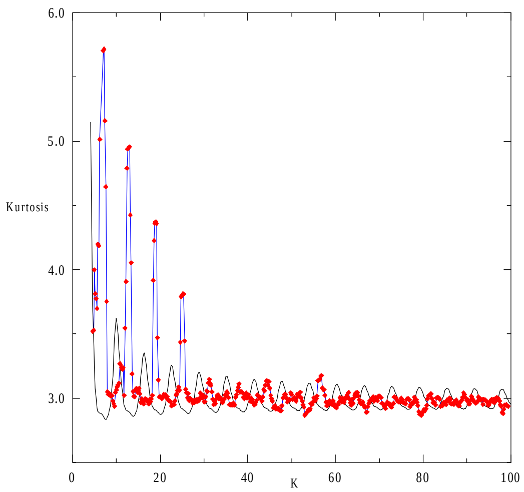

Figure 4: *We plot both the analytic and numeric result of the kurtosis versus K, with the solid line without stars is the analytic result and the solid line with stars is the numerical one.*

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