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DISPERSION EQUATION OF THE CYCLOTRON WAVES IN A TWO-DIMENSIONAL MAGNETOSPHERIC PLASMA WITH ANISOTROPIC PARTICLE ENERGY

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The energetic particles (electrons, protons, heavy ions) with anisotropic temperature can lead to a wide class of ion/electron cyclotron wave instabilities in the Earth's magnetosphere, see, e.g., Ref. [1] and bibliography therein. These instabilities should be analyzed by solving the Vlasov-Maxwell's equations, taking into account a two-dimensional nonuniformity of the geomagnetic field and bounce-resonant wave-particle interactions there, Refs. [2-4]. In this paper, the linearized Vlasov equation is solved for trapped particles with the bi-maxwellian steady-state distribution function in an axisymmetric magnetospheric plasma with circular magnetic field lines: $B(R,\phi) = B_0 R_0^3/(R^3\cos\phi)$. Here, R_0 is the radius of the Earth, R is the geocentric distance, ϕ is the geomagnetic latitude, B_0 is the magnetic field in an equatorial plane on the Earth's surface (at the point where $R = R_0, \phi = 0$). To solve the Vlasov equation we use the standard method of switching to new variables associated with the conservation integrals of energy: $v_\perp^2 + v_\parallel^2 = const$, magnetic moment: $v_\perp^2/2B = const$, and the equation of the $\mathbf{B}$-field line: $R/\cos\phi = const$. Introducing the variables $v = \sqrt{v_\parallel^2 + v_\perp^2}$, $\mu = v_\perp^2 B(L,0)/(v^2 B(L,\phi))$, $L = R/(R_0\cos\phi)$ (instead of $v_\parallel, v_\perp, R$) we seek the perturbed distribution function as

$$f(t,R,\phi,\theta,v_\parallel,v_\perp,\alpha) = \sum_s^{\pm 1}\sum_l^{\pm\infty} f_l^s(\phi,L,v,\mu)\exp(-i\omega t + im\theta + il\alpha),$$

where α is the gyrophase angle in velocity space. So that the linearized Vlasov equation for harmonics f_0^s and $f_{\pm 1}^s$ can be rewritten in the next form:

$$\sqrt{1-\frac{\mu}{\cos^4\phi}}\frac{\partial f_l^s}{\partial\phi} - is\frac{LR_0}{v}\left(\omega + \frac{l\,\omega_{co}}{L^3\cos^4\phi}\right)f_l^s = Q_l^s, \qquad l = 0, \pm 1, \tag{1}$$

where
$$Q_0^s = \frac{e}{T_\parallel}R_0 L\sqrt{1-\frac{\mu}{\cos^4\phi}}F_0 E_\parallel, \qquad E_{\pm 1} = E_n \mp iE_b,$$

$$Q_{\pm 1}^s = \frac{e}{2T_\perp}R_0 L\frac{\sqrt{\mu}}{\cos^2\phi}F_0\left[sE_{\pm 1} - i\frac{v\cos^2\phi}{\omega R_0 L}\left(\frac{T_\perp}{T_\parallel} - 1\right)\sqrt{1-\frac{\mu}{\cos^4\phi}}\frac{\partial}{\partial\phi}\frac{E_{\pm 1}}{\cos^2\phi}\right],$$

$$F_0 = \frac{N(L)}{\pi^{1.5}v_{T\parallel}v_{T\perp}^2}\exp\left\{-\frac{v^2}{v_{T\parallel}^2}\left[1-\frac{\mu}{\cos^4\phi}\left(1-\frac{T_\parallel}{T_\perp}\right)\right]\right\}, \quad v_{T\parallel}^2 = \frac{2T_\parallel}{M}, \quad v_{T\perp}^2 = \frac{2T_\perp}{M}.$$

Here, $E_\parallel, E_n, E_b$ are, respectively, the parallel, normal and binormal perturbed electric field components relative to $\mathbf{B}$; F_0 is the steady-state distribution function of plasma particles with the density N, parallel and perpendicular temperature $T_\parallel$ and $T_\perp$, respectively, charge e and mass M. By the indexes $s = \pm 1$ we differ the particles with positive and negative values of $v_\parallel = sv\sqrt{1-\mu/\cos^4\phi}$ relatively to $\mathbf{B}$. In Eq. (1) we have neglected the drift corrections assuming the wave frequency ω is much larger than the drift frequency, that is valid when $mv_{T\perp}^2 L^2/(v_{T\parallel}R_0\omega_{co}) << 1$, where $\omega_{co} = eB_0/Mc$, and m is the azimuthal wave number over θ (east-west) direction. Depending on μ, the domain of perturbed distribution functions is defined by the inequalities $L^{-4} \leq \mu \leq 1$ and $-\phi_t(\mu) \leq \phi \leq \phi_t(\mu)$, where $\pm\phi_t(\mu)$ are the local mirror points for the trapped particles at a given (by L) magnetic field line, which are defined by the zeros of parallel velocity. As a result, $\phi_t = \arccos\ \mu^{0.25}$.

After solving Eq. (1), the two-dimensional longitudinal (parallel to $\mathbf{B}$), $j_\parallel(\phi, L)$, and transverse, $j_{\pm1}(\phi, L)$, current density components can be expressed as

$$j_\parallel(\phi, L) = \frac{\pi e}{\cos^4\phi}\sum_s^{\pm 1} s \int_0^\infty v^3 \int_{L^{-4}}^{\cos^4\phi} f_0^s(\phi, L, v, \mu)\, d\mu\, dv, \tag{2}$$

$$j_l(\phi, L) = \frac{\pi e}{2\cos^4\phi}\sum_s^{\pm 1} \int_0^\infty v^3 \int_{L^{-4}}^{\cos^4\phi} \frac{\sqrt{\mu} f_l^s(\phi, L, v, \mu)}{\sqrt{\cos^4\phi - \mu}} d\mu\, dv, \qquad l = \pm 1. \tag{3}$$

Note, the normal and binormal to $\mathbf{B}$ current density components in our notation are equal to $j_n = j_1 + j_{-1}$ and $j_b = i(j_1 - j_{-1})$, respectively. In this paper, we evaluate the transverse dielectric permittivity. The longitudinal permittivity elements are derived by analogy in Ref. [5]. Note, the longitudinal permittivity in magnetospheric plasmas with an equilibrium distribution function (when $T_\parallel = T_\perp$) has been evaluated in Ref. [6] for two plasma models with dipole and circular magnetic field lines. Accounting that the trapped particles, with a given μ, execute the periodic motion with the bounce period proportional to

$$\tau_b = \tau_b(\mu) = 4\int_0^{\phi_t} \frac{\cos^2\phi}{\sqrt{\cos^4\phi - \mu}} d\phi\ ,$$

the solution of Eq. (1) (for harmonics with $l = \pm 1$) is

$$f_l^s(\phi, L, v, \mu) = \sum_{p=-\infty}^{+\infty} f_{l,p}^s(L, v, \mu) \exp\left[ip\frac{2\pi}{\tau_b}\tau(\phi) + isl\frac{R_0\omega_{co}}{L^2 v}C(\phi)\right], \tag{4}$$

where

$$\tau(\phi) = \int_0^\phi \frac{\cos^2\eta\, d\eta}{\sqrt{\cos^4\eta - \mu(\kappa)}}, \quad C(\phi) = \int_0^\phi \frac{d\eta}{\cos^2\eta\sqrt{\cos^4\eta - \mu(\kappa)}} - \frac{2E(\kappa) - K(\kappa)}{(1-2\kappa)^2\Pi(\kappa)}\tau(\phi).$$

The perturbed distribution functions, defined by Eq. (4), satisfy automatically the corresponding boundary conditions for the trapped particles, namely, the continuity of the distribution functions ($f_l^{+1} = f_l^{-1}$) at the reflection points $\pm\phi_t$. Here, a new variable $\kappa = 0.5(1 - \sqrt{\mu})$ is introduced instead of μ-variable: $\mu(\kappa) = (1 - 2\kappa)^2$. So that

$$\phi_t = \arcsin\sqrt{2\kappa}, \qquad \tau_b(\kappa) = 2\sqrt{2(1-2\kappa)}\Pi(\kappa), \qquad E(\kappa) = \int_0^{\pi/2}\sqrt{1 - \kappa\sin^2\phi}d\phi,$$

$$\Pi(\kappa) = \int_0^{\pi/2} \frac{d\phi}{(1 - 2\kappa\sin^2\phi)\sqrt{1 - \kappa\sin^2\phi}}, \qquad K(\kappa) = \int_0^{\pi/2}\frac{d\phi}{\sqrt{1-\kappa\sin^2\phi}},$$

where $K(\kappa)$, $E(\kappa)$ and $\Pi(\kappa)$ are the complete elliptic integrals of the first, second, and third kind, respectively.

After the s-summation and using the Fourier expansion of $E_{\pm1}$ over the ϕ-angle, see Eq. (7), the transverse current density component can be expressed as

$$\frac{4\pi i}{\omega} j_l(L, \phi) = \frac{\omega_{po}^2 R_0 L v_{T\parallel}}{\omega \pi^{1.5} v_{T\perp}^2 \cos^4\phi} \sum_{n'=-\infty}^{+\infty} E_l^{(n')} \sum_{p=-\infty}^{+\infty} \int_{0.5\sin^2\phi}^{(L^2-1)/2L^2} \frac{(1-2\kappa)^3 d\kappa}{\sqrt{\cos^4\phi - \mu(\kappa)}} \times$$
$$\times \int_{-\infty}^{+\infty} \frac{u^4\exp(-u^2)}{pu - Z_l(\kappa)} \exp\left(ip\frac{2\pi}{\tau_b}\tau(\phi) + il\frac{R_0\omega_{co}}{L^2 u v_{T\parallel}}C(\phi)\right) A_{p,l}^{n'}(u, \kappa) du, \tag{5}$$

where

$$\omega_{po}^2 = \frac{4\pi N e^2}{M}, \qquad u = \frac{v}{v_{T\parallel}}, \qquad Z_l(\kappa) = \frac{R_0 L \tau_b}{2\pi v_{T\parallel}}\left[\omega + l\frac{\omega_{co}}{L^3}\frac{2E(\kappa) - K(\kappa)}{(1-2\kappa)^2\Pi(\kappa)}\right],$$

$$A_{p,l}^n(u,\kappa) = \int_{-\phi_t}^{\phi_t} \exp\left[\frac{u^2\mu(\kappa)}{\cos^4\phi}\left(1 - \frac{T_\parallel}{T_\perp}\right)\right]\Phi_{p,l}^n(u,\kappa,\phi)\frac{\cos^2\phi\, d\phi}{\sqrt{\cos^4\phi - \mu(\kappa)}} +$$
$$+(-1)^p\int_{-\phi_t}^{\phi_t} \exp\left[\frac{u^2\mu(\kappa)}{\cos^4\phi}\left(1 - \frac{T_\parallel}{T_\perp}\right)\right]\Phi_{-p,l}^n(-u,\kappa,\phi)\frac{\cos^2\phi\, d\phi}{\sqrt{\cos^4\phi - \mu(\kappa)}} +$$
$$+\frac{\pi n u v_{T\parallel}}{\phi_o \omega R_0 L}\left(\frac{T_\perp}{T_\parallel} - 1\right)\left\{\int_{-\phi_t}^{\phi_t} \exp\left[\frac{u^2\mu(\kappa)}{\cos^4\phi}\left(1 - \frac{T_\parallel}{T_\perp}\right)\right]\Phi_{p,l}^n(u,\kappa,\phi)d\phi + \right.$$
$$\left. + (-1)^p\int_{-\phi_t}^{\phi_t} \exp\left[\frac{u^2\mu(\kappa)}{\cos^4\phi}\left(1 - \frac{T_\parallel}{T_\perp}\right)\right]\Phi_{-p,l}^n(-u,\kappa,\phi)d\phi\right\},$$
$$\Phi_{p,l}^n(u,\kappa,\phi) = \cos\left(\frac{\pi n}{\phi_o}\phi - p\frac{2\pi}{\tau_b}\tau(\phi) - \frac{lR_0\omega_{co}}{L^2 u v_{T\parallel}}C(\phi)\right).$$

The points $\pm\phi_0(L) = \pm\arccos(1/L)$, in Eqs. (5,7), are the beginning and the end of a given (by L) magnetic field line on the Earth's surface. As a result, the transverse current density component in an axisymmetric magnetosphere is derived by the p-summation of the bounce-resonant terms. It should be noted that the bounce-resonance conditions, $pu - Z_l(\kappa) = 0$ in Eq. (5), for the trapped particles in magnetospheric plasmas are

$$\omega + l\frac{\omega_{co}[2E(\kappa) - K(\kappa)]}{L^3\,(1-2\kappa)^2\Pi(\kappa)} = \frac{p\,\pi\,v}{R_0 L\sqrt{2(1-2\kappa)\Pi(\kappa)}}, \qquad l = 0, \pm 1. \tag{6}$$

These conditions are entirely different from the corresponding expressions in the straight magnetic field case. Of course, as in the straight magnetic field, $l = 1$ corresponds to the effective wave-electron interaction, and $l = -1$ corresponds to the wave-ion interaction. Moreover, there is no possibility to carry out the Landau integration over the particle energy $u = v/v_{T\parallel}$ (by introducing the plasma dispersion function) because the phase coefficients $A_{p,l}^{n'}$ depend on u.

To solve the two-dimensional wave equations, we should expand preliminary the perturbed values in a Fourier series over ϕ. In particular, for the transverse components of the current density, j_l, and electric field, E_l, we have:

$$\frac{j_l(L,\phi)}{\cos^2\phi} = \sum_n^{\pm\infty} j_l{}^{(n)}(L)\;\exp\left[\frac{i\pi n\phi}{\phi_0(L)}\right], \qquad \frac{E_l(L,\phi)}{\cos^2\phi} = \sum_{n'}^{\pm\infty} E_l{}^{(n')}(L)\;\exp\left[\frac{i\pi n'\phi}{\phi_0(L)}\right]. \tag{7}$$

This procedure converts the operator, representing the dielectric tensor, into a matrix whose elements are calculated independently on the solutions of Maxwell's equations. As a result,

$$\frac{4\pi i}{\omega} j_l^{(n)}(L) = \sum_{n'}^{\pm\infty} \epsilon_l^{n,n'}(L)E_l^{(n')}(L),$$

and the contribution of a given kind of plasma particles to the transverse permittivity elements, $\epsilon_l^{n,n'}(L)$, is

$$\epsilon_l^{n,n'}(L) = \frac{\omega_{po}^2 L R_0 v_{T\parallel}^3}{\omega 2\pi^{1.5} v_{T\perp}^4 \phi_o}\sum_p^{\pm\infty}\int_0^{\frac{L^2-1}{2L^2}}(1-2\kappa)^3 d\kappa\int_{-\infty}^{+\infty}\frac{u^4\exp(-u^2)}{pu - Z_l(\kappa)}D_{p,l}^n(u,\kappa)A_{p,l}^{n'}(u,\kappa)du, \tag{8}$$