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Improving the Child-Langmuir sheath model

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Abstract

A collision-free space-charge sheath formed by cold ions at a negative surface is considered. The method of matched asymptotic expansions is applied, small parameter being the ratio of the electron temperature to the sheath voltage. Boundary conditions for equations describing the bulk of the sheath are found such that the model have exponential accuracy. A physical meaning of these conditions is that the ions are accelerated in the outer section of the sheath from the Bohm velocity to twice the Bohm velocity. The model predicts sheath parameters to the accuracy of several per cent for sheath voltages exceeding $3\frac{kT_e}{e}$.

1 Introduction

A delicate question in models of electron-free space-charge sheath is that of boundary conditions at the edge. The boundary conditions used in the original Child-Langmuir model [1,2] are those of zero ion velocity and zero electric field. Some authors set the ion velocity at the sheath edge equal to the Bohm velocity (e.g., [3]). It is well-known that this is an appropriate boundary condition for a sheath with non-zero electron density, however it is unclear whether this condition is the most adequate one in the case of an electron-free sheath. In [4], a conclusion is drawn that the ions leave the quasi-neutrality region and enter the transition region with a value of the ion velocity (normalized by the Bohm velocity) $V \approx 0.806 < 1$ and leave the transition region entering the space-charge sheath with the velocity $V \approx 2.12 > 1$. Another example of boundary conditions can be found in [5].

An adequate way of formulating the conditions in question is to employ the method of matched asymptotic expansions, the large parameter being $\chi = eU/kT_e$ (here U is the voltage drop in the sheath and T_e is the electron temperature). In the framework of such an asymptotic approach, equations describing an electron-free sheath are of the exponential order of accuracy, since they are obtained by dropping the electron density term of the Poisson equation, which is exponentially small (with respect to χ). However, an algebraic-order (i.e., essentially higher) error is introduced into the model by boundary conditions. For example, the Child-Langmuir model has the accuracy $O\left(\chi^{-1/2}\right)$. There is, however, hope that an adequate treatment by means of the method of matched asymptotic expansions will reveal boundary conditions such that the accuracy of the model on the whole be exponential.

If an asymptotic model has the exponential accuracy, normally such a model is capable of giving accurate results even for values of the asymptotic parameter which are not strongly different from unity. In the particular case considered, one can hope that the model of electron-free sheath with an appropriate choice of boundary conditions would predict results with a few per cent accuracy already for surface potentials about, or even above, the floating potential.

The question of deriving such boundary conditions is considered in this work. As an example, the particular case of a collisionless sheath formed by cold ions is treated. An exact analytical solution exists in this case, which allows one to better illustrate a mathematical sense of asymptotic results and their accuracy. It should be emphasized, however, that the validity of the derived boundary conditions is not restricted to the case of a collisionless sheath: it can be shown that they are applicable also in cases when collisions play a role in the bulk of the sheath, provided that the Debye length in the quasineutral plasma is much smaller than the mean free path for collisions of ions and neutral particles.

2 The model

A mathematical problem describing a collisionless sheath formed by cold ions may be written in dimensionless variables as

$$V\frac{dV}{d\eta} = -\frac{d\Phi}{d\eta}, \qquad \frac{d^2\Phi}{d\eta^2} = -\frac{1}{V} + e^{\Phi}. \tag{1}$$

$$\eta \to \infty : \qquad V = 1, \quad \Phi = 0, \tag{2}$$

$$\eta = 0 : \quad \Phi = -\chi, \tag{3}$$

where $\chi = eU/kT_e$ is the dimensionless voltage drop in the sheath (a given positive quantity). Dimensionless variables used here are related to physical variables as

$$\eta = \frac{y}{h}, \qquad V = \frac{v_i}{u_i}, \qquad \Phi = \frac{e\phi}{kT_e}, \tag{4}$$

where $h = (\varepsilon_0 kT_e/n_s e^2)^{1/2}$, $u_i = \sqrt{kT_e/m_i}$ and all other designations are usual.

Asymptotic analysis of problem (1)-(3) was performed by means of the method of matched asymptotic expansions, χ being considered as a large parameter. It was shown that the space-charge sheath comprises two sections: a transition layer, which is an outer section of the sheath where the electron and ion densities are comparable, and an ion layer, in which the electron density is exponentially small as compared to the ion density. Below the asymptotic results are presented in a simple way, without references to the formal procedure of the method of matched asymptotic expansions.

3 Exponential-accuracy model of electron-free sheath

A system of equations describing the ion layer is obtained by neglecting the electron density term in the Poisson equation,

$$V\frac{dV}{d\eta} = -E, \qquad \frac{dE}{d\eta} = -\frac{1}{V}, \qquad \frac{d\Phi}{d\eta} = E. \tag{5}$$

A boundary condition at the surface is supplied by Eq. (3).

Problem (5), (3) may be easily solved in a general form (i.e., without specifying other boundary conditions; e.g., [4]). A convenient way to do it is as follows. Equations in Eq. (5) have first integrals

$$V - \frac{E^2}{2} = C_1, \qquad \Phi + \frac{V^2}{2} = C_2, \tag{6}$$

where C_1 and C_2 are arbitrary constants. Solving the first equation for V, substituting the result in the second equation in Eq. (5), integrating the obtained equation for the function $\eta(E)$ and taking into account boundary condition (3), one can arrive at

$$\begin{aligned}\eta \;=\; & \frac{2^{1/2}}{3}\left[\left(\sqrt{2\chi + 2C_2} - C_1\right)^{3/2} - (V - C_1)^{3/2}\right] \\ & + 2^{1/2} C_1 \left[\left(\sqrt{2\chi + 2C_2} - C_1\right)^{1/2} - (V - C_1)^{1/2}\right]. \end{aligned} \tag{7}$$

Eqs. (6) and (7) represent the desired general solution: Eq. (7) describes function $V(\eta)$, while Eq. (6) relates E and Φ to V.

One can see from Eqs. (6) and (7) that function $V(\eta)$ is decreasing and the solution cannot be extended beyond a point at which V decreases down to value $V = C_1$ or, in other words, at which $E = 0$. It is logical to consider this point as an edge of the ion layer. Thus, one arrives at the following definition of the edge of the ion layer: this is a point at which the electric field described by equations of the ion layer vanishes. It should be emphasized that what vanishes at the edge of the ion layer is not a real electric field but rather an extrapolation of a solution describing the field inside the layer.

Thus, one of boundary conditions at the edge of the ion layer for equations describing distributions inside the layer is

$$E = 0. \tag{8}$$

It follows from the asymptotic analysis that in order to obtain the exponential accuracy of the model other boundary conditions at the edge should be formulated in the following way:

$$V = 2, \quad \Phi = -\frac{3}{2}. \tag{9}$$

Note that boundary conditions (9) have quite a distinct physical sense: they imply that the ions are accelerated in the transition layer from the Bohm velocity, $V = 1$, on the plasma side of the transition layer, to twice the Bohm velocity, $V = 2$, at the boundary between the transition layer and the ion layer; voltage drop in the transition layer is $\frac{3}{2}\frac{kT_e}{e}$. One can now complete the solution by finding $C_1 = 2$, $C_2 = 1/2$.

For purposes of comparison, we note that in the framework of the Child-Langmuir model boundary conditions (9) are replaced with $V = 0$, $\Phi = 0$. A solution for this model is given by Eqs. (6) and (7) with $C_1 = C_2 = 0$.

Another model which can be used for comparison is the one in which boundary conditions at the edge of the space-charge sheath on the whole are transferred to the edge of the ion layer. The model is described by Eqs. (5), (3), (8) and (2). $C_1 = 1$ and $C_2 = 1/2$ for this model.

Accuracy of different models is illustrated by Fig. 1, in which a ratio is shown of values of the electric field at the surface given by approximate formulas to exact values. The error of the Child-Langmuir model at $\chi = 5$ (a typical value corresponding to a floating surface) is above 50%, and even at a rather high sheath voltage $\chi = 10$ the error

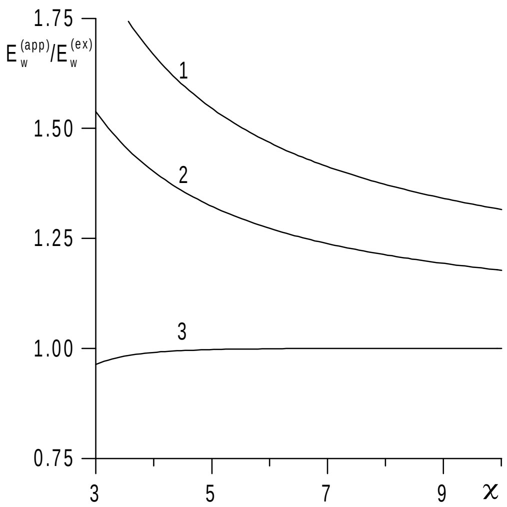

Fig. 1. Ratio of approximate values of the electric field at the surface given by different models to the exact value. 1: Child-Langmuir model. 2: ion layer-quasineutral plasma model. 3: exponential-accuracy model.

exceeds 30%. Accuracy of the ion layer-quasineutral plasma model is higher, however the improvement is not drastic. On the contrary, the error of the value given by the exponential-accuracy model does not exceed 4% in the whole range of χ considered.

4 Concluding remarks

At high values of the ratio χ of the sheath voltage to the electron temperature, the space-charge sheath comprises two sections: a transition layer and an ion layer. In a general case, an expansion describing the ion layer contains terms of algebraic orders in $1/\chi$, which originate from matching with an expansion describing the transition layer. In other words, errors of algebraic orders in $1/\chi$ are introduced in models of the ion layer by boundary conditions at the edge of the layer. Modifying these conditions, one can obtain a simple model with an exponentially small error. In this work, such a model is derived for the case of a collisionless sheath formed by cold ions.

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