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Lagrangian formulation of the transport theory for energetic ions in toroidal plasmas

Zh.N.Andrushchenko*, J.W.Edenstrasser**, & V.A.Yavorskij*

**Scientific Center "Institute for Nuclear Research", Kiev, Ukraine.*
***Institute for Theoretical Physics, University of Innsbruck, Austria.*

1. INTRODUCTION

This report is devoted to the kinetic description of the transport phenomena occurring in a magnetically confined toroidal fusion plasma by employing both the multiple timescale approach [1] and the Lagrangian formulation of kinetic theory [2]. A three-component strongly magnetized and weakly collisional plasma including the energetic ions is considered, where the parameter range of a typical fusion plasma is assumed. The drift kinetic equation applicable for the case of ions with poloidal gyro radii in the order of the plasma inhomogeneity scale length is obtained. The theory developed consistently takes into account the collisional transport effects, which may be responsible for the loss of energetic ions and should be important for the formation of the phase space distribution of confined ions in the MeV energy range. The Fokker-Planck operator is represented in a form easily accessible to a numerical treatment.

2. BASIC EQUATIONS

Energetic ions, i.e., ions with energies far above the thermal one, play a prominent role in the heating of fusion plasmas. In present-day tokamaks, minority ions accelerated by ion-cyclotron-resonance heating (ICRH) frequently reach energies in the MeV range, and deliver tens of megawatts of heating power to the bulk plasma. In a future reactor, most of the heating will be provided by fusion-generated alpha particles. The description of collisional transport processes of fast particles in toroidal plasmas is a rather difficult problem, as for a complete description of the collisional behaviour of fast particles, one should take into account the effects of slowing down, diffusive collisional transport induced by pitch-angle scattering and parallel diffusion in velocity space. However, as the oscillation times of high-energy ions in toroidal magnetic configurations are small as compared to characteristic collisional ones, one can use the gyro (and bounce) averaging that significantly simplifies the problem.

In general, the kinetic equation describing the distribution function for the particles of a given species can be written in the form

$$\frac{\partial f}{\partial t} + \dot{x}^i \frac{\partial f}{\partial x^i} = C(f) + S \,, \tag{1}$$

where x^i are arbitrary phase-space coordinates, $C(f)$ is the collisional operator and S a fast ion source. The conventional theory is essentially Eulerian in nature, working with $f=f(\mathbf{r},\mathbf{V})$, where the independent variables $\mathbf{r},\mathbf{V}$ are not constants of the orbital motion in the equilibrium field. In contrast to the Eulerian description, many of the concepts in neoclassical theory involve orbital properties and are essentially Lagrangian in nature [2]. Here we present a transport theory for the low-collisionality banana regime using what amounts to the Lagrangian picture.

In a strongly magnetized plasma the particle Larmor radius is much smaller than the gradient length scales describing the fluid behaviour of the plasma. Considering the motion in a strong, but very slowly varying magnetic field $\mathbf{B}(\mathbf{r},t)$, one can define for each point in space a set of three orthogonal unit vectors $(\boldsymbol{\tau}_1, \boldsymbol{\tau}_2, \boldsymbol{\tau}_3)$ such that

$$\boldsymbol{\tau}_1 = \mathbf{B}/B; \quad [\boldsymbol{\tau}_1\boldsymbol{\tau}_2] = \boldsymbol{\tau}_3; \quad [\boldsymbol{\tau}_2\boldsymbol{\tau}_3] = \boldsymbol{\tau}_1; \quad [\boldsymbol{\tau}_3\boldsymbol{\tau}_1] = \boldsymbol{\tau}_2. \tag{2}$$

If one expresses the particle position vector by the guiding center position and the Larmor radius vector [3]

$$\mathbf{r}_* = \mathbf{R} + \boldsymbol{\rho}, \tag{3}$$

then the Larmor rotation can be represented in the form

$$\boldsymbol{\rho} = \rho(\boldsymbol{\tau}_2 \sin\theta - \boldsymbol{\tau}_3 \cos\theta), \tag{4}$$

$$\mathbf{V} = V_{\parallel}\boldsymbol{\tau}_1 + V_{\perp}(\boldsymbol{\tau}_2 \cos\theta + \boldsymbol{\tau}_3 \sin\theta), \tag{5}$$

where ρ is the particle gyro radius $\rho = V_{\perp}\Omega^{-1}$, θ the gyro phase, and $\mathbf{V}$ the particle velocity; $V_{\parallel}$ and $V_{\perp}$ are the longitudinal and transverse components of $\mathbf{V}$. Next, the gyrophase average of any function g is defined by

$$\langle g \rangle = \frac{1}{2\pi}\oint g\, d\theta, \quad \dot{\theta} = \frac{d\theta}{dt} \approx \Omega. \tag{6}$$

As velocity variables one can use $\mathbf{V}=\{V,\xi,\theta\}$ with $\xi = V_{\parallel}/V$, and as spatial Eulerian variables one can introduce the flux coordinates $\{\Phi,\chi,\varphi\}$, where Φ is the toroidal flux, χ and φ the poloidal and toroidal angles, respectively, so that the magnetic field can be represented in the form

$$\mathbf{B} = \nabla\Phi \times \nabla\chi - \iota\nabla\Phi \times \nabla\varphi \equiv F\nabla\varphi - \iota\nabla\Phi \times \nabla\varphi, \tag{7}$$

$\iota(\Phi)$ being the rotational transform and $F(\Phi)$ the poloidal current outside the given flux surface. Using $x^i(\mathbf{r}) = x^i(\mathbf{R}+\boldsymbol{\rho}) \cong x^i(\mathbf{R}) + \boldsymbol{\rho}\cdot\nabla x^i$, new radial variables $\mathbf{R}^i$ might be introduced as follows:

$$R^1 = r^1 - \boldsymbol{\rho}\cdot\nabla r^1, \quad R^2 = r^2 - \boldsymbol{\rho}\cdot\nabla r^2, \quad R^3 = r^3 - \boldsymbol{\rho}\cdot\nabla r^3. \tag{8}$$

To simplify consideration it should be convenient to choose $\boldsymbol{\tau}_2 = \nabla\Phi/|\nabla\Phi|$. Then

$$\boldsymbol{\tau}_2 \cdot \nabla r^j = \frac{g^{1j}}{\sqrt{g^{11}}}, \ \boldsymbol{\tau}_3 \cdot \nabla r^j = \frac{1}{B\sqrt{g^{11}}}\left(g^{11}g^{2j} - g^{12}g^{1j} - \iota g^{11}g^{3j} + \iota g^{13}g^{1j}\right), \ g^{ij} = \nabla r^i \cdot \nabla r^j \tag{9}$$

The collision term in Eq.(1) is a differential operator of the form

$$C(f) = \nabla_{\mathbf{V}}\left(\mathbf{d} + \overleftrightarrow{\mathbf{D}}\nabla_{\mathbf{V}}\right) f \tag{10}$$

where $\mathbf{d}$ is the vector of the “dynamic friction force” and $\overleftrightarrow{\mathbf{D}}$ the diffusion tensor, both defined in the velocity space by

$$\mathbf{d} = \nu_S \mathbf{V}, \quad \overleftrightarrow{\mathbf{D}} = \nu_{\perp}\mathbf{V}^2\overleftrightarrow{\mathbf{I}} + (\nu_{\parallel} - \nu_{\perp})\mathbf{V}\otimes\mathbf{V}. \tag{11}$$

In Eq.(11), $\nu_S, \nu_{\perp}$ and $\nu_{\parallel}$ are the characteristic collision frequencies of slowing down, transverse and parallel diffusion and $\overleftrightarrow{\mathbf{I}}$ is the unit dyad.

If we choose for the diffusion coefficients the contravariant representation in the Eulerian variables $\mathbf{x} = (\mathbf{r}, \mathbf{V})$ defined by

$$d^i_{\mathbf{x}} = \mathbf{d}\cdot\nabla x^i, \quad D^{ij}_{\mathbf{x}} = \nabla x^i \cdot \overleftrightarrow{\mathbf{D}} \cdot \nabla x^j, \tag{12a}$$

then we simply obtain

$$\begin{gathered} d^1_{\mathbf{x}} = \nu_S V, \quad d^2_{\mathbf{x}} = 0, \quad d^3_{\mathbf{x}} = 0; \\ D^{11}_{\mathbf{x}} = \nu_{\parallel}V^2, \quad D^{22}_{\mathbf{x}} = \nu_{\perp}(1-\xi^2), \quad D^{33}_{\mathbf{x}} = D^{\theta\theta}_{\mathbf{x}} = \nu_{\perp}\frac{1}{1-\xi^2}; \quad D^{ij}_{\mathbf{x}} = 0, \quad i \neq j. \end{gathered} \tag{12b}$$

In the derivation of the gyrokinetic equation, within conventional theory one first performs for the LHS of Eq.(1) an ordering of all terms in the gyro radius and then averages over the gyro phase. In evaluating the RHS one assumes that there is no change across the Larmor radius ρ and calculates all the quantities in Eq.(10) at the guiding center position **R**. However, a more consistent way is to make a transformation from the Eulerian variables **x**=(**r**,**V**) to the Lagrangian ones **z** =(**R**,**V**), taking into account the corrections arising from the Larmor gyration and than to carry out the gyrophase averaging both the LHS of the Fokker-Planck equation and of the transformed collision operator. This approach should be essentially important for the description of the behaviour of NBI ions in spherical tokamaks and present-day stellarator devices, where the ions can have poloidal gyro radii in the order of the plasma inhomogeneity length scale. The inclusion of these additional contributions to the transport coefficients may also be important for the description of the transport phenomena at the plasma edge. The diffusion coefficients in the **z**-space are then given in the form

$$d_{\mathbf{z}}^{i} = d_{\mathbf{x}}^{j}\frac{\partial z^{i}}{\partial x^{j}}; \quad D_{\mathbf{z}}^{ij} = D_{\mathbf{x}}^{kl}\frac{\partial z^{k}}{\partial x^{k}}\frac{\partial z^{l}}{\partial x^{l}}, \tag{13}$$

with the Jacobian

$$D = \frac{\partial(\mathbf{z})}{\partial(\mathbf{x})} = \frac{\partial(\mathbf{V},\mathbf{R})}{\partial(\mathbf{V},\mathbf{r})} = \frac{\partial(z^4, z^5, z^6)}{\partial(\Phi,\chi,\varphi)} = 1, \tag{14}$$

where in the evaluation of D the Larmor oscillations are neglected. For the representation of the magnetic field in the form of Eq.(7) one arrives at

$$\sqrt{g_{\mathbf{z}}} = \sqrt{g_{\mathbf{x}}} = V^2 / F g^{33}\ . \tag{15}$$

After the transformation from the variables **x** to **z** the kinetic equation (1) can be written as

$$\frac{\partial f}{\partial t} + \dot{\mathbf{R}}\frac{\partial}{\partial \mathbf{R}} f + \dot{\mathbf{V}}\frac{\partial}{\partial \mathbf{V}} f = C(f) + S, \tag{16}$$

where the collision term in the Lagrangian coordinates $\mathbf{z} = \{\mathbf{V};\mathbf{R}\} = \{\mathbf{V}',\theta;\mathbf{R}\},\ \mathbf{V}' = \{V,\xi\}$, can be represented as a sum of the conventional collision operator and the additional contributions arising from the inclusion of finite Larmor radius effects

$$C(f) = C_{\mathbf{V}}(f) + C_{\mathbf{R}}(f), \qquad \text{with} \tag{17}$$

$$C_{\mathbf{V}}(f) = \frac{1}{\sqrt{g_{\mathbf{z}}}}\frac{\partial}{\partial \mathbf{V}'}\sqrt{g_{\mathbf{z}}}\left(d_{\mathbf{z}}^{\mathbf{V}'} + D_{\mathbf{z}}^{\mathbf{V}'\mathbf{V}'}\frac{\partial}{\partial \mathbf{V}'}\right) f + \frac{\partial}{\partial \theta}\left(D_{\mathbf{z}}^{\theta\theta}\frac{\partial}{\partial \theta}\right) f, \tag{18a}$$

$$\begin{aligned} C_{\mathbf{R}}(f) = {} & \frac{1}{\sqrt{g_{\mathbf{z}}}}\frac{\partial}{\partial \mathbf{V}'}\sqrt{g_{\mathbf{z}}}\left(D_{\mathbf{z}}^{\mathbf{V}'\mathbf{R}}\frac{\partial}{\partial \mathbf{R}}\right) f + \frac{\partial}{\partial \theta}\left(D_{\mathbf{z}}^{\theta\mathbf{R}}\frac{\partial}{\partial \mathbf{R}}\right) f + \\ & + \frac{1}{\sqrt{g_{\mathbf{z}}}}\frac{\partial}{\partial \mathbf{R}}\sqrt{g_{\mathbf{z}}}\left(d_{\mathbf{z}}^{\mathbf{R}} + D_{\mathbf{z}}^{\mathbf{R}\mathbf{V}'}\frac{\partial}{\partial \mathbf{V}'} + D_{\mathbf{z}}^{\mathbf{R}\theta}\frac{\partial}{\partial \theta} + D_{\mathbf{z}}^{\mathbf{R}\mathbf{R}}\frac{\partial}{\partial \mathbf{R}}\right) f. \end{aligned} \tag{18b}$$

Recalling the gyrophase average of Eq.(6) we can define both, the distribution function and the diffusion coefficients by

$$g(\mathbf{V};\mathbf{R}) = g(\mathbf{V}',\theta;\mathbf{R}) = \langle g(\mathbf{V}';\mathbf{R})\rangle + \tilde{g}(\mathbf{V}',\theta;\mathbf{R}). \tag{19}$$

Since the characteristic frequencies satisfy the following hierarchy of inequalities

$$\Omega >> \nu_S >> \nu_{\parallel}, \nu_{\perp}, \tag{20}$$

the multiple timescale approach [1,2] may be applied. In order to find $\tilde{f} = \tilde{f}(\langle f\rangle)$, one can start from the Fokker-Planck equation (16) and collect all terms of the same order in ρ.

Following the approach of [4-6], we will keep only the lowest order contribution arising from the inclusion of the Larmor radii that results in

$$\tilde{\dot{\mathbf{R}}} \cdot \frac{\partial}{\partial \mathbf{R}} \langle f \rangle + \Omega \frac{\partial}{\partial \theta} \tilde{f} = 0, \qquad (21)$$

where both terms are of order unity. The part of the distribution function connected with fast Larmor gyration may be easily found in the form

$$\tilde{f} = -\int d\theta \frac{\tilde{\dot{\mathbf{R}}}}{\Omega} \cdot \frac{\partial}{\partial \mathbf{R}} \langle f \rangle = -\boldsymbol{\rho} \cdot \frac{\partial}{\partial \mathbf{R}} \langle f \rangle, \qquad (22)$$

being in the agreement with the classical results of Hinton & Hazeltine [4]. After the gyrophase averaging, we have reduced Eq.(16) to the drift-kinetic equation:

$$\frac{\partial \langle f \rangle}{\partial t} + \mathbf{V}_{gc} \frac{\partial}{\partial \mathbf{R}} \langle f \rangle + \dot{\mathbf{V}}' \frac{\partial}{\partial \mathbf{V}'} \langle f \rangle = \langle C_{\mathbf{V}}(f) \rangle + \langle C_{\mathbf{R}}(f) \rangle + \langle S \rangle, \qquad (23)$$

where $\mathbf{V}_{gc} = \langle \dot{\mathbf{R}} \rangle = \mathbf{V}_{\parallel} + \mathbf{V}_D$ is the guiding center velocity and $\mathbf{V}_D$ is the drift velocity. The collision operators can then be represented in the form

$$\langle C_{\mathbf{V}}(f) \rangle = \frac{1}{\sqrt{g_{\mathbf{z}}}} \left(\frac{\partial}{\partial V} \sqrt{g_{\mathbf{z}}} \left(d_{\mathbf{z}}^{1} + D_{\mathbf{z}}^{11} \frac{\partial}{\partial V} \right) + \frac{\partial}{\partial \xi} \sqrt{g_{\mathbf{z}}} \left(D_{\mathbf{z}}^{22} \frac{\partial}{\partial \xi} \right) \right) \langle f \rangle = C_{\mathbf{V}'}(\langle f \rangle), \qquad (24)$$

$$\langle C_{\mathbf{R}}(f) \rangle = \left\langle \frac{1}{\sqrt{g_{\mathbf{z}}}} \frac{\partial}{\partial \mathbf{R}} \sqrt{g_{\mathbf{z}}} d_{\mathbf{z}}^{\mathbf{R}} \tilde{f} \right\rangle = \frac{1}{\sqrt{g_{\mathbf{z}}}} \frac{\partial}{\partial R^{i}} \sqrt{g_{\mathbf{z}}} \left(\nu_S \frac{\rho^2}{2} g^{ij} \frac{\partial}{\partial R^{j}} \right) \langle f \rangle. \qquad (25)$$

It should be pointed out that the result of Eq.(25) is due to the fact that the slowing down rate ν_S for fast particles significantly exceeds the pitch angle and parallel diffusion rates $\nu_{\perp}$ and $\nu_{\parallel}$. Thus, the consistent approach in the derivation of the drift kinetic equation has resulted in additional contributions to the transport coefficients.

3. CONCLUSIONS

The drift kinetic equation applicable for the case of ions with poloidal gyroradii of the order of the plasma inhomogeneity scale length is obtained, where additional contributions to the transport coefficients are found. The present investigations may be important for the description of the behaviour of NBI ions in spherical tokamaks and present-day stellarator devices and the charged fusion products in future tokamak and stellarator reactors, as well as for the transport phenomena at the plasma edge.

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