Incompressible pressure determinations

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Abstract Certain unresolved ambiguities surround pressure determinations for incompressible flows, both Navier-Stokes and magnetohydrodynamic (MHD). For uniform-density fluids with standard Newtonian viscous terms, taking the divergence of the equation of motion leaves a Poisson equation for the pressure to be solved. But Poisson equations require boundary conditions. For the case of rectangular periodic boundary conditions, pressures determined in this way are unambiguous. But in the presence of "no-slip" rigid walls, the equation of motion can be used to infer both Dirichlet and Neumann boundary conditions on the pressure P, and thus amounts to an over-determination. This has occasionally been recognized as a problem, and numerical treatments of wall-bounded shear flows usually have built in some relatively *ad hoc* dynamical recipe for dealing with it, often one which appears to "work" satisfactorily. Here we consider a class of solenoidal velocity fields which vanish at no-slip walls, have all spatial derivatives, but are simple enough that explicit analytical solutions for P can be given. Satisfying the two boundary conditions separately gives two pressures, a "Neumann pressure" and a "Dirichlet pressure" which differ non-trivially at the initial instant, even before any dynamics are implemented. We compare the two pressures, and find that in particular, they lead to different volume forces near the walls. This suggests a reconsideration of no-slip boundary conditions, in which the vanishing of the tangential velocity at a no-slip wall is replaced by a local wall-friction term in the equation of motion.

In fluid mechanics, meaningful confrontation between experiment and theory begins when theory starts to take fluid boundary conditions and their implementation seriously. Generally speaking, fluids do what they do because of what is done to them at their boundaries. One may ask if this truism does not apply also to plasmas. It may be taken as indicating the incomplete maturity of fusion theory that the majority of theoretical plasma calculations have given little or no attention to implementation of boundary conditions, however unrealistic.

In recent years [1-3], we have addressed several MHD confinement problems theoretically with an emphasis on satisfying toroidal boundary conditions (however oversimplified) at material walls and including a full range of transport coefficients in the dynamics, sometimes with disquieting results. For example, flows of a kind that do not characterize straight-cylinder models of MHD seem to be necessary for force balance, even in equilibrium.

Here, we focus on one small aspect of this problem, determination of pressure in incompressible fluids and MHD systems in which there are flows present. Incompressibility, if not always an accurate representation of laboratory situations, greatly simplifies dynamical fluid and MHD calculations because it makes an equation of state and a dynamical equation for the internal energy or temperature unnecessary. However, it raises certain difficulties associated with pressure determinations that do not have a ready answer.

The equation of mechanical motion for MHD in the simplest version reads:

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = \frac{\mathbf{j} \times \mathbf{B}}{c} - \nabla P + \nu \nabla^2 \mathbf{v}, \tag{1}$$

where $\mathbf{v} = \mathbf{v}(\mathbf{x}, t)$ is the fluid velocity vector, $\mathbf{B}$ is the magnetic field, P is the pressure normalized to the mass density, assumed uniform and constant. The current density $\mathbf{j} = c\nabla \times \mathbf{B}/4\pi$, where c is the speed of light (Gaussian units), and ν is the kinematic viscosity, also assumed uniform and constant. (1) is to be supplemented, of course, by Faraday's law, Ohm's law (assuming a finite scalar conductivity), and the statements that both $\mathbf{B}$ and $\mathbf{v}$ are solenoidal ($\nabla \cdot \mathbf{v} = 0, \nabla \cdot \mathbf{B} = 0$). $\nabla \cdot \mathbf{v} = 0$ functions in effect as an equation of state; when we take the divergence of (1), it implies that $\nabla^2 P = -\nabla \cdot (\mathbf{v} \cdot \nabla \mathbf{v} - \mathbf{j} \times \mathbf{B}/c)$, which is a Poisson equation for P, a solution of which will determine P as a (non-local) functional of $\mathbf{v}$, $\mathbf{j}$, and $\mathbf{B}$. The Navier-Stokes fluid case is specializable from the MHD description by going through and setting all terms in $\mathbf{B}$ and $\mathbf{j}$ equal to zero. The difficulty we will illustrate is most simply exhibited by considering the Navier-Stokes case first. Consider the case of "no-slip" rigid material walls, where it is demanded that $\mathbf{v}$ should go to zero, conventionally. If we approach the walls with (1), the two terms on the left should go to zero, leaving as a boundary condition on P the relation $\nabla P = \nu \nabla^2 \mathbf{v} + \mathbf{j} \times \mathbf{B}/c$. However, a moment's reflection shows that this amounts to an overdetermination of the pressure P. Solutions to Poisson's equation are well known to be determined uniquely by giving either the normal component of ∇P at the boundary or by giving P itself there, but giving both amounts to an overdetermination, and cannot be done. These are called "Neumann" and "Dirichlet" boundary conditions, respectively. Clearly, specifying the two tangential components of the gradient of P amounts to giving Dirichlet conditions as well as the Neumann conditions specified by the normal component.

Perusal of the fluids/MHD literature reveals only two cases where the Poisson equation for P seems to have been straightforwardly useful at determining P: one-dimensional steady shear flows, such as pipe flow or Couette flow, and rectangular periodic boundary conditions such as are used in the theory of homogeneous turbulence. Elsewhere, there are a variety of relatively ad hoc and only partially justified recipes for demanding that a non-zero tangential velocity at a no-slip wall shall become small as a function of time as a consequence of an ad hoc procedure concocted for the purpose (e.g. [4], [5]). Molecular dynamics studies of the near-wall region in molecular gases have revealed a more complicated situation than will comfortably fit into a macroscopic hydrodynamics or magnetohydrodynamics [6]. However, many numericists

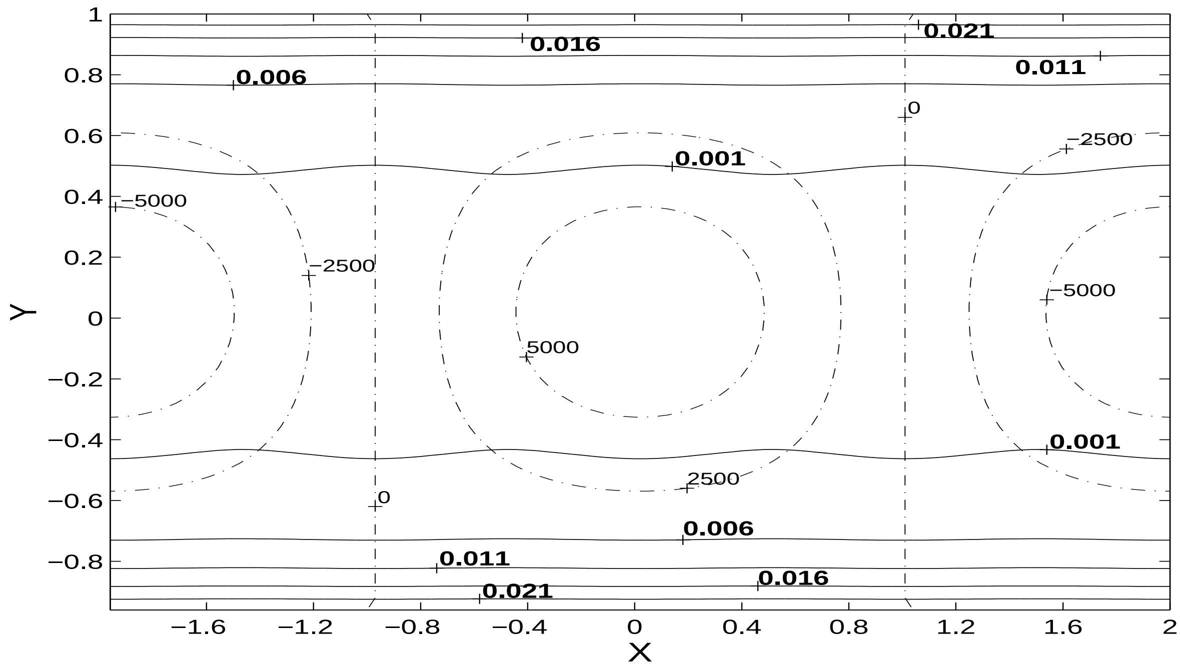

Figure 1. Streamlines (dot-dash line) using ψ from (2) with $k = \pi/2$, $\lambda = 2.642$ $A_{k\lambda} = .349$ and $C_{k\lambda} = 5000$.
Normalized mean square pressure gradient difference r (solid countour), with $Re = 2290$. Note that the fractional difference between the two values of ∇P is significant only near the wall.

have designed and run incompressible, wall-bounded, shear-flow dynamical numerical codes whose results do not seem to be at noticeable variance with experimental results, even though they inevitably contain mathematically unsupported steps somewhere in their core when it comes to enforcing no-slip boundary conditions. We illustrate the difficulty by considering a class of two dimensional velocity fields derived from the stream function,

$$\psi(x, y) = C_{k\lambda} \cos(kx)[\cos(\lambda y) + A_{k\lambda} \cosh(ky)] \tag{2}$$

where $\mathbf{v} = \nabla\psi \times \hat{\mathbf{e}}_z$, and the constants λ and $A_{k\lambda}$ are determined numerically, so that both components of $\mathbf{v}$ vanish at $y = \pm a$ to any desired accuracy. We illustrate, in figure 1, a streamline plot (dot-dash lines) given by choosing $k = \pi/2$, $\lambda = 2.642$ and $A_{k\lambda} = .349$, in units of $a = 1$. This is an example drawn from a much larger class of functions related to Chandrasekhar-Reid functions [7]. The flow is periodic in the x-direction, has all spatial derivatives, is solenoidal, and perfectly satisfies no-slip boundary conditions at the walls at $y = \pm a$. We may ignore any MHD complications or decorations in the interest of making our point, and focus on the pure Navier-Stokes case, substituting only $\mathbf{v}$ obtained from (2) into the "source" term on the right hand side of the Poisson equation, in order to see what pressure P it will demand.

The source term contains only terms which are products and sums of exponentials of kx, λy, and ky, so that despite some tedious algebra, it is elementary to generate an inhomogeneous solution for P. To this may be added any solution of Laplace's equation with the same x periodicity. The combination may be chosen to satisfy either the Neumann condition on P, or the Dirichlet condition, but not both. We thus generate at $t = 0$, before any dynamics are advanced, two initial pressures, a "Neumann pressure" P_N and a "Dirichlet pressure" P_D, which are not identical and may be compared. In figure 1, a fractional measure of the difference between P_N and P_D is exhibited as a contour plot of the scalar ratio $r \equiv (\nabla P_D - \nabla P_N)^2 / \langle(\nabla P_N)^2\rangle$. It initially increases with $Re(\equiv (\langle \mathbf{v}^2\rangle/(k^2+\lambda^2))^{1/2}/\nu)$, approaching a maximum of about 2% near the wall for $Re \gtrsim 10$.

This unsatisfactory nature of the incompatibility between incompressibility-determined pressures and no-slip boundary conditions may suggest, even at this late date, the desirability of a modification of the latter. One recipe implemented in a rotating MHD computation some years ago [8, 9] involved a replacement of the no-slip requirement by an addition to the right hand side of (1) of a wall-friction term $-\mathbf{v}/\tau(\mathbf{x})$, where $1/\tau(\mathbf{x})$ is a function of position which vanishes in the interior but rises to large values in the immediate vicinity of the wall, in a region smaller in dimension than that of any boundary layer one might hope to resolve. This permits the wall to absorb momentum from the fluid and forces the tangential velocity to a low value, but does not force it exactly to zero: a finite "slip" velocity remains. This recipe seemed to perform adequately in the situation in which it was used, and may merit further more stringent tests in the future in both MHD and Navier-Stokes cases. These are planned.

A more detailed presentation of this work will be published later [10]. (One of us (D.C.M.) was supported by hospitality at the Eindhoven University of Technology in the Netherlands).

[1] Montgomery D and Shan X 1994 *Comments on Plasma Phys. and Control. Fusion* **15**, 315.

[2] Bates J W and Montgomery D 1998 *Phys. Plasmas* **5**, 2649.

[3] Montgomery D, Bates J W and Kamp L P 1999 *Plasma Physics and Controlled Fusion* **41**, A507.

[4] Gresho P M 1991 *Annu. Rev. Fluid Mech.* **23**, 413–453.

[5] Canuto C, Hussaini M Y, Quarteroni A and Zang T A 1988 *Spectral Methods in Fluid Mechanics.* Springer.

[6] Thompson P A and Trolan S M 1997 *Nature* **389**, 360.

[7] Chandrasekhar S 1961 *Hydrodynamic and Hydromagnetic Stability* Oxford Univ. Press, p.634

[8] Shan X and Montgomery D C 1994a *Phys. Rev. Letters* **73**, 1624.

[9] Shan X and Montgomery D C 1994b *J. Plasma Physics* **52**, 113.

[10] Kress B T and Montgomery D C 2000 *J. Plasma Phys.* (to appear).