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Theory of three-wave coupling in a warm magnetized plasmas.

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1. Introduction

The three-wave matrix elements, which determine the quadratic non-linear current, are widely used in the theory of a wave scattering in plasmas as well as in weak turbulence theory. In the case of non-magnetized homogeneous plasmas the explicit expressions for them are expressed in terms of linear plasma conductivity tensor and could be found in text-books [1, 2]. On contrary in the case of magnetized plasma only complicated general integral expressions are available. These expressions take an explicit simple form in the «cold plasma» limit, when the MHD approximation is applicable for description of all three interacting waves. The corresponding so called MHD model for non-linear current is often used for investigation of non-linear and scattering phenomena in magnetized plasma. It's often applied also to interaction of small-scale waves for which the cold plasma model is not satisfactory. Some background for such a wide use of the MHD expressions has been provided recently in [4], where the expressions resembling the standard MHD results were derived for the case where the MHD approximation is valid for the scattered wave only.

The present paper is devoted to the detailed analyses of the general expression for the non-linear induce charge. It is simplified under assumption that the frequency of one of interacting waves is lower then two others. No assumptions concerning the wavelength of the waves are done. It is shown that the standard MHD approximation overestimates the non-linear induced charge (or current) for interaction of small-scale waves and domain of its validity is given. The obtained results are applied for the interpretation of the UHR scattering from Ion Bernstein Waves in FT-1 tokamak [5].

2. Kinetic model

In collisionless plasma the dynamic of the electron momentum distribution $f\left(\vec{p},\vec{r},t\right)$ is governed by the Vlasov equation

$$\frac{\partial f}{\partial t}+\vec{v}\frac{\partial f}{\partial \vec{r}}+\frac{e}{m}\left(\vec{E}+\frac{\vec{v}\times\vec{B}}{c}\right)\frac{\partial f}{\partial \vec{v}}=0 \tag{1}$$

We seek for the solution of this equation representing f as a series with perturbation variable $\delta \propto v_E/v_t$, v_E is oscillating velocity and $v_t=\sqrt{T_e/m_e}$ - thermal electron velocity:

$$f = f^{(0)}(\vec{p})+ f^{(1)}(\vec{p},r,\vec{t})+ f^{(2)}(\vec{p},r,\vec{t})+.... \tag{2}$$

Here $f^{(0)}$ is Maxwellian distribution function and $f^{(n)}(\vec{p},\vec{r},t)\propto\delta^{(n)}$.

The linear correction term $f^{(1)}(\vec{p},\vec{r},t)$ is defined by first order quantity equation:

$$\hat{L}f^{(1)}=-\vec{F}^{(1)}\frac{\partial f}{\partial \vec{v}} \tag{3}$$

Here we have introduced the linear operator $\hat{L}$:

$$\hat{L}=\frac{\partial}{\partial t}+\vec{v}\frac{\partial}{\partial \vec{r}}+\frac{e}{m}\left(\vec{E}+\frac{\vec{v}\times\vec{B}^{(0)}}{c}\right)\frac{\partial}{\partial \vec{v}} \tag{4}$$

In a case of two independent harmonic waves $\left\{\left(\omega(a),\vec{k}(a)\right)\text{and}\left(\omega(b),\vec{k}(b)\right)\right\}$ existing in plasma the first order correction to distribution function is $f^{(1)}=f^{(1a)}+f^{(1b)}$. The correction terms $f^{(1a)}$, $f^{(1b)}$ are independent solutions of the equation (3), which are well known from the linear theory.

Due to non-linear interaction between two independent harmonic waves the charge

$$\rho^{(2\sigma)}=e\int f^{(2\sigma)}d^3v \tag{5}$$

is induced possessing the frequency and wave-vector

$$\omega(\sigma)=\omega(a)+\omega(b) \quad \vec{k}(\sigma)=\vec{k}(a)+\vec{k}(b) \tag{6}$$

The quadratic correction for the distribution function $f^{(2\sigma)}$ is described by second order equation:

$$\hat{L}f^{(2\sigma)}=-\vec{F}^{(1a)}\frac{\partial f^{1b}}{\partial \vec{v}}-\vec{F}^{(1b)}\frac{\partial f^{1a}}{\partial \vec{v}} \tag{7}$$

This ordinary inhomogeneous differential equation has the following solution:

$$f^{(2\sigma)}=\frac{1}{\omega_{ce}}e^{-i\beta_\perp(\sigma)\sin(\phi+\theta(\sigma))}\int_0^\infty e^{-i\alpha(\sigma)\tau+i\beta_\perp(\sigma)\sin(\phi-\tau+\theta(\sigma))}\Theta(\phi-\tau,v_\perp,v_z)d\tau \tag{8}$$

Here $\beta_\perp(\sigma)=v_\perp k_\perp(\sigma)/\omega_{ce}$, $\alpha(\sigma)=(v_z k_z(\sigma)-\omega(\sigma))/\omega_{ce}$, $\phi=\arccos\left(v_x/v_\perp\right)$,

$\theta(\sigma)=\arccos\left(k_x(\sigma)/k_\perp(\sigma)\right)$, $\Theta(\phi,v_\perp,v_z)=-\vec{F}^{(1a)}\dfrac{\partial f^{(1b)}}{\partial \vec{v}}$

We've omitted here the small second term in (7). Inserting (8) into the expression (6) we find

$$\rho^{(2\sigma)}=\frac{e}{\omega_{ce}}\int\left[e^{-i\beta_\perp(\sigma)\sin(\phi+\theta(\sigma))}\int_0^\infty e^{-i\alpha(\sigma)\tau}e^{i\beta_\perp(\sigma)\sin(\phi-\tau+\theta(\sigma))}\Theta(\phi-\tau)d\tau\right]d\vec{v} \tag{9}$$

Here $d\vec{v}=v_\perp dv_\perp dv_z d\phi$.

After integrating over ϕ, $v_\perp$ and v_z, we get

$$\rho^{(2\sigma)}=\frac{i}{4\pi}\frac{\omega_{pe}^2}{\omega_{ce}}\frac{eE_k^{(1a)}E_l^{(1b)}}{mk_z(b)v_t^2}\int_0^\infty d\tau\cdot\Psi_k(\tau)\Xi_l(\tau)\exp\left(-\left(\frac{\lambda_z^T(\sigma)\tau}{\sqrt{2}}\right)^2+i\frac{\omega(\sigma)}{\omega_{ce}}\tau\right) \tag{10}$$

Here vectors $\Psi_k(\tau)$ and $\Xi_l(\tau)$ have following components respectively:

$$\left\{i\frac{k_\perp(\sigma)}{\omega_{ce}}\left(\sin\theta(\sigma)-\sin\left(\tau+\theta(\sigma)\right)\right),i\frac{k_\perp(\sigma)}{\omega_{ce}}\left(\cos\theta(\sigma)-\cos\left(\tau+\theta(\sigma)\right)\right),-i\frac{k_z}{\omega_{ce}}\tau\right\}$$

$$\left\{0;i\frac{Z(\tilde{\mu})}{\tilde{\mu}}\frac{\partial}{\partial\beta_\perp^T(b)}A\left(\beta_\perp^T(b),\zeta^T\right);-\left(1-\frac{\mu}{\tilde{\mu}}Z(\tilde{\mu})\right)A\left(\beta_\perp^T(b),\zeta^T\right)\right\},\ \tilde{\mu}=\mu+i\lambda_z^T(\sigma)\tau$$

$$A\left[\beta_\perp^T,\zeta^T\right]\equiv 2\int_0^\infty \exp\left(-t^2\right)J_0\left(\beta_\perp^T t\right)J_0\left(\zeta^T t\right)tdt\,,\lambda_z^T=\frac{\sqrt{2}k_z v_t}{\omega_{ce}},\beta_\perp^T=\frac{\sqrt{2}k_\perp v_t}{\omega_{ce}}$$

$$\zeta^T=\sqrt{\beta_\perp^T(a)^2+\beta_\perp^T(\sigma)^2-2\beta_\perp^T(a)\beta_\perp^T(\sigma)\cos(\tau+\theta(\sigma))}\,,Z\left(\tilde{\mu}\right)=X\left(\tilde{\mu}\right)-iY\left(\tilde{\mu}\right),$$

$$X\left(\tilde{\mu}\right)=2\tilde{\mu}\exp\left(-\tilde{\mu}^2\right)\int_0^{\tilde{\mu}}\exp\left(t^2\right)dt\,,\ Y\left(\tilde{\mu}\right)=\sqrt{\pi}\tilde{\mu}\exp\left(-\tilde{\mu}^2\right),\ \mu=\frac{\omega(b)}{\omega_{ce}\lambda_z^T(b)}.$$

3. Analytical limits of nonlinear induced charge density.

1. First we consider the following case: $\beta_\perp^T(a)\beta_\perp^T(\sigma)<<1$**,** $\left[\dfrac{k_z(b)v_t}{\omega_{ce}}\right]<<1$

After some calculation we find $\rho^{(2\sigma)}=\rho^{(2\sigma)(0)}+\rho^{(2\sigma)(1)}$

$$\rho^{(2\sigma)(0)}=\frac{i}{4\pi}\frac{n^{(1b)}}{n_o}\mu_k^{(0)}(\sigma)E_k^{(1a)}, \tag{11}$$

$n^{(1b)}=\int d\vec{v}f^{(1b)}$, $\mu_k^{(0)}(\sigma)=\dfrac{4\pi}{i}\dfrac{k_i(\sigma)\ \sigma_{ik}(\sigma)}{\omega(\sigma)}$, $\sigma_{ik}(\sigma)$- the «cold» tensor of the plasma conductivity calculated for $\omega(\sigma)$.

$$\rho^{(2\sigma)(1)}=\frac{i}{4\pi}\frac{E_k^{(1a)}}{N_0}\left[\tilde{n}(b,\sigma)\mu_k^{(1)}(\sigma)+\breve{n}(b,\sigma)\left(\mu_k^{(0)}(\sigma)\cos\theta(\sigma)-\mu_k^{(2)}(\sigma)\right)\right].$$

Here $\tilde{n}(b,\sigma)=\dfrac{eE_k^{(1b)}\tilde{W}_k^{(1,1)}}{mv_t^2k_z(b)}N_0$, $\breve{n}(b,\sigma)=\dfrac{eE_k^{(1b)}\tilde{W}_k^{(1,2)}}{mv_t^2k_z(b)}N_0$, $\mu_k^{(1)}(\sigma)=-i\dfrac{\partial}{\partial u}\mu_k^{(0)}(\sigma)$,

$$\mu_k^{(2)}(\sigma)=-i\frac{\omega_{pe}^2}{\omega_{ce}^2}\{\Xi1;\Xi2;\Xi3\},$$

$$\tilde{W}_l^{(1,1)}=\left\{0;i\beta_\perp^T(b)A_0';A_0\right\}\cdot\left(Z'(\mu)-\frac{Z(\mu)}{\mu}\right)\cdot\frac{2ik_z(\sigma)\ k_z(b)v_t^2}{\omega(b)\omega_{ce}},$$

$$\tilde{W}_l^{(1,2)}=\left\{0;i\frac{Z(\mu)}{\mu}\frac{\beta_\perp^T(b)}{2}A_0;\left(1-Z(\mu)\right)A_0\right\}\cdot\frac{\beta_\perp^T(a)\beta_\perp^T(\sigma)}{2}.$$

The first term in the equation (11) coincides with the MHD approximation expression [3]. However in contradiction to [3] it dominates in expression for $\rho^{(2\sigma)}$ only for

$\beta_\perp^T(\sigma)\beta_\perp^T(a)$ □ 1 and not for $\beta_\perp^T(\sigma)<<1$.

2. Coming to opposite limiting case of small-scale wave interaction, let us suppose

$$1 << \beta_\perp^T(b), \beta_\perp^T(\sigma), \beta_\perp^T(a), 1 \ll \frac{\beta_\perp^T(a)\beta_\perp^T(\sigma)\sin\theta(\sigma)}{\left|\beta_\perp^T(b)\right|} \text{ and } \frac{k_z(\sigma)v_t}{\omega_{ce}} \cdot \frac{k_z(b)v_t}{\omega(b)} \ll 1.$$

These conditions can hold in particular for scattering of electron Bernstein wave after linear conversion in the Upper Hybrid Resonance.

Using saddle point method and assuming $\frac{\omega(\sigma)-m\omega_{ce}}{k_z(\sigma)v_t} \gg 1, m \in Z$ we find final expression

$$\rho^{(2\sigma)} = \frac{i}{4\pi}\frac{n^{(1b)}}{n_o}\mu_i^{(3)}E_i^{(1a)}, \text{ where } \quad \mu_x^{(3)} = (2\pi)^{1/2}\frac{i\omega_{pe}}{v_t}\frac{k_\perp(b)}{k_\perp(a)}\frac{1}{1-\exp i\dfrac{\omega(\sigma)}{\omega_{ce}}} \tag{12}$$

4. Summary

The analyses of the induced non-linear charge density, based on condition $\omega(b) \ll \omega(\sigma), \omega(a)$, have confirmed validity of the MHD expressions in the long scale limit. However unlike the prediction of [3] the domain of applicability of simple MHD expression (11) is shown to be limited to the case $k_\perp(a)k_\perp(\sigma)\rho_e^2 \ll 1$ and not $k_\perp(\sigma)\rho_e \ll 1$.

As it is seen from the small scale limit for the non-linear induced charge density (12), the MHD expression (11) overestimates the $\rho^{(2\sigma)}$ by a factor of $k_\perp(\sigma)\rho_e \gg 1$. Such a suppression of the non-linear charge is probably the reason of a rather low level of signal observed in experiments on tokamak FT -1 [4] in the UHR backscattering experiments.

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