Plasma Rotation: Transition from linear to nonlinear dynamics

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Abstract

The standard theoretical approach to plasma dynamics, consisting of a separate study of a static equilibrium state and of the perturbations of this background, is invalidated by present heating and exhaust removal techniques in fusion experiments. Neutral beam heating causes the plasma to spin and divertors for particle and energy extraction result in the supersonic flow of a narrow plasma layer on the outside of the tokamak. These kinds of flow present an enormous challenge, both with respect to the construction of stationary flow patterns, with possible transitions to super-Alfvénic flows, and with respect to the investigation of the different oscillations and new types of instabilities. In this paper, (1) the modifications of the linear magnetohydrodynamic waves and instabilities of plasmas with background rotation are summarized; (2) the nonlinear stationary flow patterns that occur in laboratory and astrophysical plasmas when the background speed traverses the full range of critical speeds are analysed; (3) it is shown how new solutions bridge the gap between insights obtained from the linear and the nonlinear analyses.

1 Waves in plasma with background flow

The standard approach for the study of MHD waves and instabilities in tokamaks is to split the problem in that of *static equilibrium + linear waves and instabilities*, as described by the equations $\mathbf{j} \times \mathbf{B} = \nabla p$ and $\mathbf{F}(\boldsymbol{\xi}) = -\rho\omega^2\boldsymbol{\xi}$. This approach has been successfully followed during 40 years of intensive research. However, astrophysical plasmas are *never* in static equilibrium. They are all dominated by flows that are usually *transsonic*: they cross all critical MHD speeds and exhibit shocks. In recent tokamak research, significant toroidal and poloidal flows are also created. Hence, *plasmas with background flow* becomes a relevant common research theme of laboratory and astrophysical plasmas.

For the study of MHD waves in plasmas with background flow, again, the standard approach may be followed. Now, the equations describing the stationary background equilibrium ($\mathbf{v} \neq 0$) are much more complicated since none of the ideal MHD equations in trivially satisfied. Also, the study of the linear waves and instabilities requires the substantially more involved spectral problem formulated by Frieman and Rotenberg [1]. In this paper, we are concerned about this split in equilibrium and perturbations.

Spectral theory for static equilibria is a very powerful instrument, analogous to quantum mechanics for atomic systems. E.g. MHD spectroscopy [2] could offer accurate information on plasmas, similar to that obtained in helioseismology. The short-wavelength limit determines the spectral structure resulting in *three singular continuous spectra*,

$$\text{slow}: \ \{\omega_S^2(x)\}\,, \quad \text{Alfvén}: \ \{\omega_A^2(x)\}\,, \quad \text{fast}: \ \omega_F^2 \ (=\infty)\,. \tag{1}$$

For equilibria with flow the local Doppler shifted frequencies, $\tilde{\omega} \equiv \omega - \mathbf{k}\cdot\mathbf{v}$, give rise to *new continuous spectra*:

$$\Omega_S^{\pm} = \pm\omega_S + \mathbf{k}\cdot\mathbf{v}\,, \quad \Omega_A^{\pm} = \pm\omega_A + \mathbf{k}\cdot\mathbf{v}\,, \quad \Omega_F^{\pm} = \pm\infty\,. \tag{2}$$

Moreover, the problem is no longer self-adjoint: *overstable modes* are possible.

In general, the spectra with flow become extremely involved. However, for weak inhomogeneity, the three types of waves remain distinguishable: there are now forward and backward slow, Alfvén, and fast waves, with each subspectrum clustering at its own continuum given by Eqs. (2). The continua are organised around the central frequency range $\{\Omega_0(x)\}$, where $\Omega_0 \equiv \mathbf{k}\cdot\mathbf{v}$ is the local Doppler shift. In contrast to spectra for static equilibria, which are symmetric about $\omega = 0$, the spectrum is now asymmetric.

For strong inhomogeneity, the continua may become unstable, new gaps appear, etc. A significant example are the new global modes induced by toroidal flow found by van der Holst et al. [3]. These modes occur at low-frequency. Due to toroidal flow, a tiny gap in the continuum at marginal stability in the static case is Doppler shifted and much widened. In this new gap, a global Alfvén wave appears. This mode should provide a useful diagnostic for MHD spectroscopy in the presence of background plasma flow.

Let us now turn to our point of concern. Perturbations of the flow propagate along space-time manifolds called *characteristics*. The MHD group diagram represents a snapshot of the spatial part of such a characteristic. Note that the Lagrangian time derivative

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla \tag{3}$$

introduces temporal phenomena (waves and instabilities) through $\partial/\partial t$, whereas the spatial derivative ∇ becomes relevant in plasmas with a background flow. The spatial part of the characteristics then betrays their wave-like origin so that *linear waves and nonlinear stationary equilibria are no longer a separate issue*. Hence, rather than studying the waves, we should *reconsider the background equilibrium state itself*.

2 Transsonic MHD Flow

To appreciate the intricacies of transsonic MHD flow, consider 2D translation symmetric stationary equilibria [4]. Toroidicity is included in forthcoming publications [5], [6]. The essential variables are the poloidal magnetic field $\mathbf{B}_p = \mathbf{e}_z \times \nabla\psi$ and velocity $\mathbf{v}_p = \mathbf{e}_z \times \nabla\chi$, derived from streamfunctions $\psi(x,y)$ and $\chi(\psi)$, and the square of *the poloidal Alfvén Mach number*, $\mu(x,y) \equiv \rho v_p^2/B_p^2 = \chi'^2/\rho$. The remaining physics is described by five arbitrary flux functions, which combine to three: $\Pi_{1,2,3}(\psi)$. The basic problem then is, for a given choice of the Π's, to determine the poloidal distribution of $\psi(x,y)$ & $\mu(x,y)$.

This problem may be cast in a variational principle stating that the stationary states are obtained by minimizing the following Lagrangian:

$$\delta \int \mathcal{L}\, dV = 0\,, \quad \mathcal{L} \equiv \tfrac{1}{2}(1-\mu)|\nabla\psi|^2 - W(\psi,\mu)\,, \quad W \equiv \frac{\Pi_1(\psi)}{\mu} - \frac{\Pi_2(\psi)}{\gamma\mu^{\gamma}} + \frac{\Pi_3(\psi)}{1-\mu}\,. \tag{4}$$

This yields a nonlinear PDE for the determination of the magnetic flux $\psi(x,y)$ and an algebraic equation, the Bernoulli equation, for the Mach number $\mu(x,y)$:

$$\nabla\cdot[\,(1-\mu)\,\nabla\psi\,] + \frac{\partial W}{\partial\psi} = 0\,, \qquad \tfrac{1}{2}|\nabla\psi|^2 + \frac{\partial W}{\partial\mu} = 0\,. \tag{5}$$

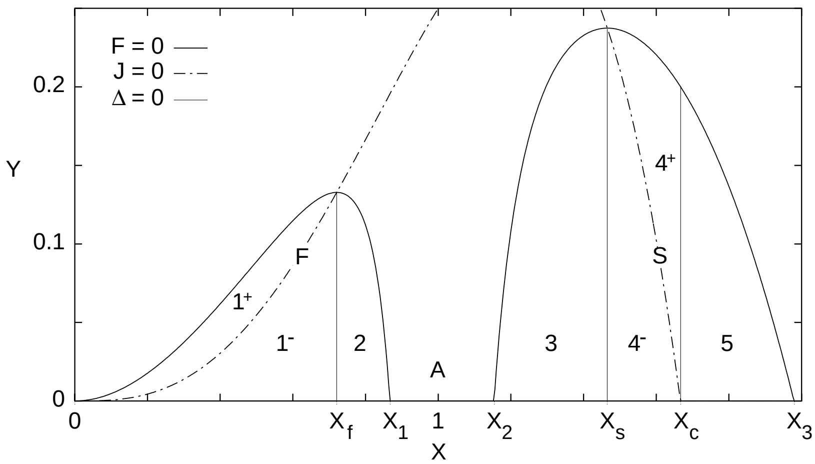

Figure 1: Four main flow domains due to the Alfvén gap & fast and slow limiting lines.

Note that the variable $\mu(x, y)$ is to be obtained from Eq. (5)b and then inserted in Eq. (5)a to appear in front of the highest derivative of the flux equation.

Reduction to tractable size is achieved by choosing *a master profile* $\pi \equiv \psi^{2-2/\lambda}$, so that $\Pi_1 = \pi(\psi)$, $\Pi_2 = A\,\pi(\psi)$, $\Pi_3 = B\,\pi(\psi)$, and assuming *self-similarity* in polar coordinates: $\mu^{-1} = X(\theta)$, $\psi = r^\lambda Y(\theta)$. *A system of ODEs* for X and Y is then obtained:

$$\frac{dX}{d\theta} = \pm\frac{H}{J}\sqrt{2F}\,, \qquad \frac{dY}{d\theta} = \pm\sqrt{2F}\,. \tag{6}$$

The different *flow regimes* follow from algebraic conditions in X and Y, viz. the fast and slow Bernoulli Boundaries ($F = 0$), the condition for Limiting Line characteristics ($J = 0$), and the condition for occurrence of characteristics ($\Delta < 0 : \mathcal{E}$ / $\Delta \geq 0 : \mathcal{H}$).

These conditions produce a phase diagram with four main flow regimes (Fig. 1):

$$\mathcal{H}_{ff}\,(1^+) \quad [\![F]\!] \quad \mathcal{H}_f\,(1^-)\,,\ \mathcal{E}_f\,(2) \quad [\![A]\!] \quad \mathcal{E}_s\,(3)\,,\ \mathcal{H}_s\,(4^-) \quad [\![S]\!] \quad \mathcal{H}_{ss}\,(4^+)\,,\ \mathcal{E}_{ss}\,(5)\,.$$

The Bernoulli condition $F \geq 0$ gives rise to fast and slow flow domains, the $J = 0$ singularity splits these domains in superfast (ff) and fast (f), and in slow (s) and subslow (ss) subdomains, whereas the $\Delta = 0$ condition gives rise to the vertical lines separating elliptic ($\mathcal{E}$) and hyperbolic ($\mathcal{H}$) regions.

Consider the flow patterns that correspond to the trajectories in the X–Y phase plane (not shown in Fig. 1). These trajectories either intersect the limiting line or not. In the latter case, poloidal flows are obtained that are periodic. In the first case, they show the very peculiar behavior in the physical x–y plane shown in Fig. 2(a). A slow flow ($\mathcal{E}_s\,, \mathcal{H}_s$), with increasing speed in the bottom part of the figure and decreasing speed in the upper part, hits a limiting line and is then 'reflected' to manifest subslow flow ($\mathcal{E}_{ss}\,, \mathcal{H}_{ss}$) in the same sector of the poloidal plane. Apparently, smooth crossing of the limiting line in the X–Y phase plane does not correspond to acceptable solutions in the physical x–y plane.

Our conclusion should be that the occurrence of limiting line characteristics indicates that singular discontinuous flows are to be considered, i.e. we should permit *shocks*. We then arrive at the following shock conditions:

– The flux variable Y and the toroidal field parameter B should be continuous;

– The Mach variable X and the entropy parameter A may jump, but A increases;
Elimination of A_2 results in the *distilled jump condition* $f(\hat{X}_1, \hat{X}_2, \hat{Y}, A_1) = 0$ and *entropy condition* $g(\hat{X}_1, \hat{X}_2, \hat{Y}, A_1) \geq 0$, where the index 1/2 refers to ahead/behind the shock;

– The variables should stay within the *Bernoulli boundaries*: $F(\hat{X}_1, \hat{Y}, A_1, B) \geq 0$.

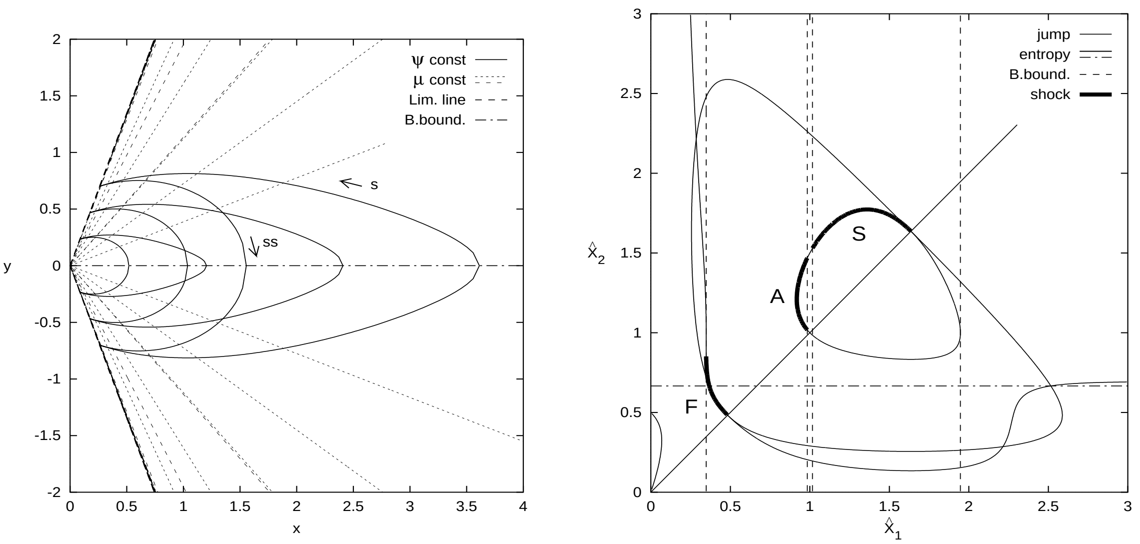

Figure 2: (a) Flow 'reflected' by the limiting line. (b) Fast, Alfvén, and slow shocks.

To find the permitted kinds of shocks we plot f in the $\hat{X}_1$-$\hat{X}_2$ plane, for given $\hat{Y}$ and A_1, and cut out the forbidden entropy and Bernoulli parts. An illustrative result of this procedure is shown in Fig. 2(b): just three pieces of the jump curve remain. These pieces precisely correspond to singular behavior with the features of slow, Alfvén, and fast wave polarizations: The three shocks permit the solutions to jump across the limiting lines. They may be considered to be the *non-linear counterparts of the three linear MHD waves.*

3 Conclusions

• The structure of the spectrum of MHD waves hinges on the three continua where the perturbations become singularly localised.
• With flow, this structure becomes very complex due to space-dependent Doppler shifts.
• In tokamaks, wide gaps with *global modes induced by toroidal flow* have been found [3].
• With rotation, the singular structure transfers to the equilibrium state: it exhibits precisely three types of 'singularities' where the variables are discontinuous.
• Consequently, *linear waves and non-linear stationary equilibria are no longer a separate issue in magnetohydrodynamics*.
• Realistic solutions for transsonic astrophysical and laboratory plasma flows have been constructed by means of state-of-the-art numerical methods [7], [6].

References

[1] Frieman, E. and Rotenberg, M., *Rev. Mod. Phys.* **32**, 898 (1960).
[2] Goedbloed, J.P., Huysmans, G.T.A., Holties, H.A., Kerner, W., and Poedts, S., *Phys. Controlled Fusion* **35**, B277 (1993).
[3] van der Holst, B., Beliën, A.J.C., and Goedbloed, J.P., *Phys. Rev. Lett.* **84**, 2865 (2000); *Phys. Plasmas*, submitted (2000); also these proceedings.
[4] Goedbloed, J.P., Lifschitz, A., *Phys. Plasmas* **4**, 3544 (1997).
[5] Goedbloed, J.P., Lifschitz, A., to be published (2000).
[6] Beliën, A.J.C., Goedbloed, J.P., van der Holst, B., these proceedings.
[7] Keppens, R, Goedbloed, J.P., *Ap. J.* **530**, 1036 (2000).